

A distance exponent for Liouville quantum gravity

Ewain Gwynne, Nina Holden, and Xin Sun

Massachusetts Institute of Technology

Abstract

Let $\gamma \in (0, 2)$ and let h be the random distribution on \mathbb{C} which parametrizes a γ -Liouville quantum gravity (LQG) cone. Also let $\kappa = 16/\gamma^2 > 4$ and let η be a whole-plane space-filling SLE_κ curve independent from h and parametrized by γ -quantum mass with respect to h . We study a family $\{\mathcal{G}^\epsilon\}_{\epsilon>0}$ of planar maps associated with (h, η) , which we call the *LQG structure graphs*, and which we conjecture converges in the scaling limit in the Gromov-Hausdorff topology to a metric on the γ -LQG cone. In particular, \mathcal{G}^ϵ is the graph whose vertices are segments of the form $\eta([(k-1)\epsilon, k\epsilon])$ for $k \in \mathbb{Z}$, with two such segments connected by an edge if and only if they share a non-trivial boundary arc. Due to the peanosphere description of SLE-decorated LQG due to Duplantier, Miller, and Sheffield (2014), the graph \mathcal{G}^ϵ can equivalently be expressed as an explicit functional of a certain correlated two-dimensional Brownian motion, so can be studied without any reference to SLE or LQG.

We prove that there is an exponent $\chi > 0$ for which the expected graph distance between generic points in the subgraph of \mathcal{G}^ϵ corresponding to the segment $\eta([0, 1])$ is of order $\epsilon^{-\chi+o_\epsilon(1)}$, and this distance is extremely unlikely to be larger than $\epsilon^{-\chi+o_\epsilon(1)}$. In the special case when $\gamma = \sqrt{2}$, we show that the diameter of this subgraph of \mathcal{G}^ϵ is of order $\epsilon^{-\chi+o_\epsilon(1)}$ with high probability. We also prove non-trivial upper and lower bounds for the cardinality of a graph-distance ball of radius n in \mathcal{G}^ϵ which are consistent with the prediction of Watabiki (1993) for the Hausdorff dimension of LQG.

Contents

1	Introduction	2
1.1	Overview	2
1.2	Basic notations	5
1.3	Definition of the structure graph	5
1.4	Main results	6
1.5	Related works	9
1.6	Discrete intuition	11
1.7	Outline	13
2	Preliminaries	13
2.1	Background on LQG and SLE	13
2.2	Basic properties of the structure graph	14
2.3	Brownian motion estimates	18
3	Quantitative distance bounds	20
3.1	Lower bound for the diameter	20
3.2	KPZ formula for expected Minkowski dimension	22
3.3	Lower bound for the cardinality of a ball	26
4	Expected diameter of a cell conditioned on its boundary lengths	32
4.1	Conditioning on just the endpoints	32
4.2	Estimates for conditioned Brownian bridge	33
4.3	Proof of Proposition 4.1	36

5	Existence of an exponent via subadditivity	36
5.1	A variant of Fekete's subadditivity lemma	38
5.2	Conditioned concentration bound	39
5.3	Proof Proposition 5.1	43
5.4	Proof of Proposition 5.2	45
6	General distance estimates	46
6.1	Expected distance between uniformly random or fixed points	46
6.2	Non-dyadic cell counts	48
6.3	Proof of Theorems 1.12 and Theorem 1.15	50
7	Distance to the lower boundary for $\gamma = \sqrt{2}$	50
7.1	Re-rooting	51
7.2	Distance conditioned on empty lower boundary	52
7.3	Proof of Proposition 7.1	55
8	Distance to the union of three boundary arcs for $\gamma = \sqrt{2}$	57
8.1	Distances in an excursion away from the running infimum	58
8.2	Proof of Proposition 8.1	62

1 Introduction

1.1 Overview

Let $\gamma \in (0, 2)$ and let $D \subset \mathbb{C}$ be a simply connected domain. A γ -Liouville quantum gravity (LQG) surface is the surface parametrized by D whose Riemannian metric tensor is (heuristically speaking) given by

$$e^{\gamma h} dx \otimes dy \tag{1.1}$$

where h is some variant of the Gaussian free field [She07] on D and $dx \otimes dy$ is the Euclidean metric tensor. Liouville quantum gravity is a natural model of a continuum random surface. One reason for this is that LQG is the conjectured scaling limit of various random planar map models, the most natural discrete random surfaces. The case when $\gamma = \sqrt{8/3}$ corresponds to pure gravity, which is the scaling limit of uniform random planar maps. Other values of γ arise from random planar maps weighted by the partition function of some statistical mechanics model, e.g., the uniform spanning tree ($\gamma = \sqrt{2}$), the Ising model ($\gamma = \sqrt{3}$), or a bipolar orientation ($\gamma = \sqrt{4/3}$). For many such models, it is expected that the scaling limit of the statistical mechanics model on the planar map is described by an SLE $_{\kappa}$ -type curve [Sch00] or a family of such curves, independent from the LQG surface, for $\kappa = \gamma^2$ or $\kappa = 16/\gamma^2$.

Since h is a distribution, or generalized function, and is not well-defined pointwise, the formula (1.1) does not make rigorous sense. However, one can rigorously construct the volume form associated with the metric (1.1), which should be a regularized version of $e^{\gamma h(z)} dz$, where dz is the Euclidean volume form. This was accomplished in [DS11], where it was shown that several different regularization procedures for $e^{\gamma h(z)} dz$ converge to the same limiting measure μ_h , the γ -quantum area measure induced by h . See also [RV14] and the references therein for a more general theory of regularized random measures. The procedure used in [DS11] also allows one to define a length measure ν_h on certain curves in $D \cup \partial D$ (including ∂D and independent SLE $_{\kappa}$ -type curves for $\kappa = \gamma^2$).

A major problem in the study of LQG is to make sense of (1.1) as a random metric (distance function). This has recently been accomplished in the special case when $\gamma = \sqrt{8/3}$ by Miller and Sheffield in the series of works [MS16c, MS15c, MS15a, MS15b, MS16a, MS16b], using a random growth process called quantum Loewner evolution. For certain special types of quantum surfaces defined in [DMS14], the resulting metric space is isometric to a certain *Brownian surface*, a random metric space which locally looks like the Brownian map [Le 13, Mie13] and which arises as the scaling limit of certain uniform random planar maps. In particular, the quantum sphere is isometric to the Brownian map, the $\sqrt{8/3}$ -quantum cone is isometric to the Brownian plane [CL14], the quantum disk is isometric to the Brownian disk [BM15] and the $\sqrt{8/3}$ -quantum wedge

is isoemtric to the Brownian half-plane [GM16b, BMR16]. In the case when $\gamma \neq \sqrt{8/3}$, the problem of constructing a LQG metric remains open.

Another major problem is to determine the Hausdorff dimension of the γ -LQG metric, assuming that it exists. In the case when $\gamma = \sqrt{8/3}$ it is known that this dimension is 4 [Le 07]. For general γ it is predicted by Watabiki [Wat93] that the dimension d_γ of γ -LQG is a.s. given by

$$d_\gamma = 1 + \frac{\gamma^2}{4} + \frac{1}{4} \sqrt{(4 + \gamma^2)^2 + 16\gamma^2}. \quad (1.2)$$

There have been several works which support the Watabiki prediction. The authors of [AB14] perform numerical simulations using the discrete GFF which agree with the formula (1.2). In [MS16c, Section 3.3], the authors give an alternative non-rigorous derivation of (1.2) using so-called quantum Loewner evolution processes. The works [MRVZ14, AK14] prove upper and lower bounds for the Liouville heat kernel. If one assumes a certain relationship between two exponents (which the authors of the mentioned papers are not able to verify), these estimates suggest upper and lower bounds for the LQG dimension; this will be discussed further in Section 1.5. There is also a related quantity for LQG, called the *spectral dimension*, which is expected to be equal to 2 for all values of γ [ANR⁺98]. This prediction is confirmed in the context of the Liouville heat kernel in [AK14].

In contrast to the above results, the recent work [DG16b] proves estimates for several natural approximations of the γ -LQG metric (different from the approximations considered in this paper) for small values of γ which appear to contradict the Watabiki prediction; c.f. Section 1.5.

If the dimension of LQG is d_γ , it is expected that the diameter (with respect to the graph distance) of a random planar map with n edges which converges in the scaling limit to LQG is typically of order $n^{1/d_\gamma + o_n(1)}$. Hence computing the dimension of LQG is expected to be equivalent to computing the tail exponent for the diameter of a random planar map in the LQG universality class.

The goal of this article is to present some small progress toward the above two problems. Miller and Sheffield's approach in the case $\gamma = \sqrt{8/3}$ does not have a direct generalization to other values of γ , since it relies on special symmetries which are unique to $\gamma = \sqrt{8/3}$. Instead, we will use a different approach based on the peanosphere construction of [DMS14], which makes sense for all values of $\gamma \in (0, 2)$ and which we will now describe.

A *peanosphere* is a random pair (M, η) consisting of a topological space M and a space-filling curve η on M (with a specified parametrization) which is constructed from a correlated two-sided two-dimensional Brownian motion (see Figure 5). A peanosphere has a natural volume measure, which is defined by the condition that η traces one unit of mass in one unit of time.

The main result of [DMS14] is that there is a canonical (up to rotation) way to embed a peanosphere into \mathbb{C} in such a way that the volume measure is mapped to the γ -quantum area measure corresponding to a particular type of γ -LQG surface called a γ -quantum cone, represented by a distribution h ; and the curve η is mapped to a space-filling variant of SLE_κ which is independent from h . The correlation of the peanosphere Brownian motion is given by $-\cos(\pi\gamma^2/4)$ (see [GHMS16] for a proof of this fact when $\gamma < \sqrt{2}$). The two coordinates of this Brownian motion give the net change in the quantum lengths of the left and right sides of η relative to time 0, respectively, so are denoted by L_t and R_t for $t \in \mathbb{R}$. We write $Z_t = (L_t, R_t)$ for the peanosphere Brownian motion.

In this article, we will study a family of planar maps $\{\mathcal{G}^\epsilon\}_{\epsilon>0}$, called the *LQG structure graphs* associated with the pair (h, η) , which we expect converges in the scaling limit in the Gromov-Hausdorff sense (when equipped with their graph distances) to a metric on the surface parametrized by h . The vertices of the structure graph \mathcal{G}^ϵ are the elements of $\epsilon\mathbb{Z}$ (or $\epsilon\mathbb{Z} \cap (0, 1]$ in the finite-volume case). Two such vertices x_1 and x_2 are connected by an edge if the corresponding “cells” $\eta([x_1 - \epsilon, x_1])$ and $\eta([x_2 - \epsilon, x_2])$ intersect along a non-trivial boundary arc. Equivalently, in terms of the Brownian motion $Z = (L, R)$, $x_1, x_2 \in \epsilon\mathbb{Z}$ are adjacent if and only if either

$$\left(\inf_{t \in [x_1 - \epsilon, x_1]} L_t \right) \vee \left(\inf_{t \in [x_2 - \epsilon, x_2]} L_t \right) \leq \inf_{t \in [x_1, x_2 - \epsilon]} L_t \quad (1.3)$$

or the same holds with R in place of L . Consequently, the LQG structure graphs can be defined using only Brownian motion, without any reference to LQG or SLE. In fact, most of the theorems and proofs in this

paper can be phrased entirely in terms of Brownian motion. See Figure 1 for an illustration of the structure graphs.

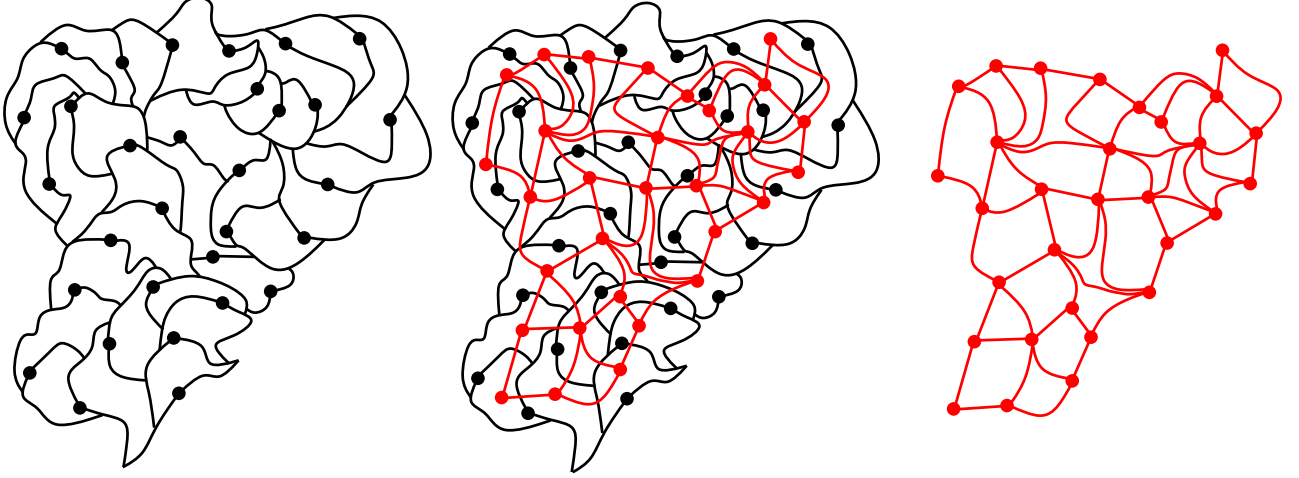


Figure 1: Left: the set $\eta([0, T])$ for some $T > 0$, divided into cells $\eta([x - \epsilon, x])$ for $x \in (0, T]_{\epsilon\mathbb{Z}}$. Here $\kappa \geq 8$, so η does not make “bubbles”. Middle: the restricted structure graph $\mathcal{G}^\epsilon|_{(0, T]}$ (Definition 1.8) is the graph whose vertex set is $(0, T]_{\epsilon\mathbb{Z}}$, with $x, y \in (0, T]_{\epsilon\mathbb{Z}}$ connected by an edge if and only if the corresponding cells share a non-trivial boundary arc. This graph has a natural embedding into \mathbb{C} , where each vertex is mapped to the corresponding cell. This embedded graph is shown in red. Right: the structure graph without the underlying collection of cells.

The LQG structure graphs are also studied in [DMS14, Section 10] (but not referred to as such), where they are used to prove that the peanosphere Brownian motion Z a.s. determines the pair (h, η) . In [MS15b, Section 1.1], the structure graphs are suggested as a possible approach for constructing a metric on LQG. These graphs can also be viewed as a potential approach to [DMS14, Question 13.1], since they depend only on the peanosphere Brownian motion Z .

One a priori reason to expect that the LQG structure graphs should yield a metric on LQG in the scaling limit comes from comparison to discrete models. Indeed, as we will explain in Section 1.6, the LQG structure graph is a natural continuum analogue of the graph metric on certain random planar maps which belong to the LQG universality class, including FK planar maps [She16b], bipolar-oriented planar maps [KMSW15], and activity-weighted planar maps [GKMW16]. Another reason to expect that the structure graphs yield a metric on LQG in the scaling limit is as follows. If we restrict attention to a single coordinate of Z (say L) and define two elements of $\epsilon\mathbb{Z}$ to be adjacent if (1.3) holds, then it is easy to see that the resulting metric spaces converge in the scaling limit to the continuum random tree (CRT) [Ald91a, Ald91b, Ald93] constructed from L . Hence the structure graph construction is a natural two-dimensional generalization of an approximation to the CRT.

We will prove that there is an exponent $\chi = \chi(\gamma) > 0$ such that the expected distance between two generic points of $(0, 1] \cap \epsilon\mathbb{Z}$ in the internal graph metric of \mathcal{G}^ϵ on $(0, 1] \cap \epsilon\mathbb{Z}$ is of order $\epsilon^{-\chi+o_\epsilon(1)}$ and the distance between two such points is extremely unlikely to be greater than $\epsilon^{-\chi+o_\epsilon(1)}$. In the special case when $\gamma = \sqrt{2}$ (in which case the coordinates of the peanosphere Brownian motion are independent), we obtain stronger results which imply in particular that the diameter of the internal graph metric of \mathcal{G}^ϵ on $(0, 1] \cap \epsilon\mathbb{Z}$ is of order $\epsilon^{-\chi+o_\epsilon(1)}$ with probability tending to 1 as $\epsilon \rightarrow 0$.

We also prove non-trivial upper and lower bounds for the cardinality of a graph-distance ball of radius n in \mathcal{G}^ϵ , which we expect should scale like $n^{d_\gamma+o_n(1)}$, where d_γ is the dimension of γ -LQG. Hence our bounds can be interpreted as upper and lower bound for d_γ . As we will explain in Section 1.5 below, these bounds for d_γ are consistent with both the Watabiki prediction (1.2) and the estimates of [DG16b] and sharper than what can be obtained from [MRVZ14, AK14] (although there is presently no direct rigorous connection between our results and those of [MRVZ14, AK14, DG16b]).

Acknowledgements We thank Jian Ding, Jason Miller, and Scott Sheffield for helpful discussions. E.G. was supported by the U.S. Department of Defense via an NDSEG fellowship. N.H. was supported by a doctoral research fellowship from the Norwegian Research Council.

1.2 Basic notations

Here we record some basic notations which we will use throughout this paper.

Notation 1.1. For $a < b \in \mathbb{R}$ and $c > 0$, we define the discrete intervals $[a, b]_{c\mathbb{Z}} := [a, b] \cap (c\mathbb{Z})$ and $(a, b)_{c\mathbb{Z}} := (a, b) \cap (c\mathbb{Z})$.

Notation 1.2. If a and b are two quantities, we write $a \preceq b$ (resp. $a \succeq b$) if there is a constant C (independent of the parameters of interest) such that $a \leq Cb$ (resp. $a \geq Cb$). We write $a \asymp b$ if $a \preceq b$ and $a \succeq b$.

Notation 1.3. If a and b are two quantities which depend on a parameter x , we write $a = o_x(b)$ (resp. $a = O_x(b)$) if $a/b \rightarrow 0$ (resp. a/b remains bounded) as $x \rightarrow 0$ (or as $x \rightarrow \infty$, depending on context). We write $a = o_x^\infty(b)$ if $a = o_x(b^s)$ for each $s > 0$ (resp. $s < 0$) as $x \rightarrow 0$ (resp. $x \rightarrow \infty$). The regime we are considering will be clear from the context.

Unless otherwise stated, all implicit constants in \asymp , \preceq , and \succeq and $O_x(\cdot)$ and $o_x(\cdot)$ errors involved in the proof of a result are required to depend only on the auxiliary parameters that the implicit constants in the statement of the result are allowed to depend on.

Remark 1.4. For $\epsilon \rightarrow 0$, we allow errors of the form $o_\epsilon(1)$ to be infinite for large values of ϵ . In particular, the statement “ $f(\epsilon) \geq \epsilon^{o_\epsilon(1)}$ ” for some function $f : (0, \infty) \rightarrow [0, \infty)$ means that for each $\zeta > 0$, there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0]$ we have $f(\epsilon) \geq \epsilon^\zeta$.

We next introduce some notation concerning graphs.

Definition 1.5. For a graph G , we write $\mathcal{V}(G)$ for the set of vertices of G and $\mathcal{E}(G)$ for the set of edges of G .

Definition 1.6. Let G be a graph and let $n \in \mathbb{N} \cup \{\infty\}$. A *path* of length n in G is a sequence $P = \{P(i)\}_{i \in [1, n]_{\mathbb{Z}}} \subset \mathcal{V}(G)$ such that $P(i)$ is connected to $P(i-1)$ by an edge in G for each $i \in [2, n]_{\mathbb{Z}}$. We write $|P| = n$ for the length of P . We say that P is *simple* if P does not visit any vertex of G more than once.

Definition 1.7. For a graph G and vertices $x, y \in \mathcal{V}(G)$, we write $\text{dist}(x, y; G)$ for the graph distance from x to y in G , i.e. the infimum of the lengths of paths in G joining x to y . For a set $V \subset \mathcal{V}(G)$, we write

$$\text{diam}(V; G) := \sup_{x, y \in V} \text{dist}(x, y; G).$$

We write $\text{diam}(G) := \text{diam}(\mathcal{V}(G); G)$. For $x \in \mathcal{V}(G)$ and $r \geq 0$, we define the *ball of radius r centered at x in G* by

$$\mathcal{B}_r(x; G) := \{y \in \mathcal{V}(G) : \text{dist}(x, y; G) \leq r\}.$$

For many of the statements in this paper, the particular choice of graph G used in Definition 1.7 will be important. We note that if G' is a subgraph of G , then one has $\text{dist}(x, y; G') \geq \text{dist}(x, y; G)$ for each $x, y \in \mathcal{V}(G)$.

1.3 Definition of the structure graph

In this subsection we define the planar maps which we will study in this paper. See Section 2.1.1 for a review of the objects involved. Let $\gamma \in (0, 2)$ and $\kappa = 16/\gamma^2 > 4$. Let $(\mathbb{C}, h, 0, \infty)$ be a γ -quantum cone [DMS14, Section 4.3]. Let η be a whole-plane space-filling SLE $_\kappa$ from ∞ to ∞ which is independent from h , parametrized by γ -quantum area with respect to h , and satisfying $\eta(0) = 0$ [MS13, Sections 1.2.3 and 4.3]. Let $Z = (L, R)$ be the quantum boundary length process for η , so that by [DMS14, Theorem 1.13]

(if $\kappa \in (4, 8]$) or [GHMS16, Theorem 1.1] (if $\kappa > 8$) the process Z is a correlated two-sided two-dimensional Brownian motion with correlation $-\cos(4\pi/\kappa)$.

For $\epsilon > 0$, let \mathcal{G}^ϵ be the graph whose vertex set is $\epsilon\mathbb{Z}$, with two vertices $x_1, x_2 \in \epsilon\mathbb{Z}$ are joined by an edge in \mathcal{G}^ϵ if and only if $\eta([x_1 - \epsilon, x_1])$ and $\eta([x_2 - \epsilon, x_2])$ share a non-trivial (i.e., positive length) boundary arc. Note that this means that $x \in \epsilon\mathbb{Z}$ corresponds to the time interval $[x - \epsilon, x]$ for η . The peanosphere construction [DMS14, Theorem 1.13] implies that a.s. \mathcal{G}^ϵ can equivalently be defined by declaring that $x_1, x_2 \in \epsilon\mathbb{Z}$ with $x_1 < x_2$ are adjacent if and only if there is an $s_1 \in [x_1 - \epsilon, x_1]$ and an $s_2 \in [x_2 - \epsilon, x_2]$ such that either

$$\inf_{r \in [s_1, s_2]} L_r = L_{s_1} = L_{s_2} \quad \text{or} \quad \inf_{r \in [s_1, s_2]} R_r = R_{s_1} = R_{s_2}. \quad (1.4)$$

Existence of s_1 and s_2 such that (1.4) holds is also equivalent to the condition that

$$\begin{aligned} \left(\inf_{s \in [x_1 - \epsilon, x_1]} L_s \right) \vee \left(\inf_{s \in [x_2 - \epsilon, x_2]} L_s \right) &\leq \inf_{s \in [x_1, x_2 - \epsilon]} L_s \quad \text{or} \\ \left(\inf_{s \in [x_1 - \epsilon, x_1]} R_s \right) \vee \left(\inf_{s \in [x_2 - \epsilon, x_2]} R_s \right) &\leq \inf_{s \in [x_1, x_2 - \epsilon]} R_s. \end{aligned} \quad (1.5)$$

Vertices of \mathcal{G}^ϵ correspond to sets of the form $\eta([x - \epsilon, x])$ for $x \in \epsilon\mathbb{Z}$. We refer to such sets as *cells*. See Figure 1 for an illustration of the graph \mathcal{G}^ϵ . We will make frequent use of the following notation.

Definition 1.8. For a set $A \subset \mathbb{R}$, we write $\mathcal{G}^\epsilon|_A$ for the subgraph of \mathcal{G}^ϵ whose vertex set is $\epsilon\mathbb{Z} \cap A$, with two vertices adjacent if and only if they are adjacent in \mathcal{G}^ϵ .

Remark 1.9. If $\kappa \in (4, 8)$, then for $x_1, x_2 \in \epsilon\mathbb{Z}$ with $x_1 < x_2$, it holds with positive probability that $\eta([x_1 - \epsilon, x_1])$ and $\eta([x_2 - \epsilon, x_2])$ intersect, but do not share a non-trivial boundary arc. This can be the case, e.g., if η closes off a bubble (i.e. disconnects some open set from ∞) at a time in $[x_2 - \epsilon, x_2]$ whose boundary it began tracing at a time in $[x_1 - \epsilon, x_1]$ and which it does not finish filling in until after time $x_2 + \epsilon$. In this case the x_1 and x_2 are *not* adjacent in \mathcal{G}^ϵ . If $\kappa \geq 8$, the path η hits the points on its outer boundary for the second time in the opposite order that these points were hit for the first time, so does not form bubbles. Hence in this case (1.4) is a.s. equivalent to the condition that $\eta([x_1 - \epsilon, x_1]) \cap \eta([x_2 - \epsilon, x_2]) \neq \emptyset$.

The main reason for our interest in the graphs \mathcal{G}^ϵ is the following conjecture, further motivation for which is provided in Section 1.6.

Conjecture 1.10. *The appropriately normalized graph distances in the graphs \mathcal{G}^ϵ converge in the scaling limit to a metric on γ -LQG. This metric is the scaling limit in the Gromov-Hausdorff topology of any random planar map model which converges to SLE-decorated γ -LQG in the peanosphere sense, including those studied in [She16b, KMSW15, GKMW16]. In the case when $\gamma = \sqrt{8/3}$, the limiting metric coincides with the one in [MS15b, MS16a, MS16b] (which is itself isometric to the Brownian map [Le 14, Mie09]). Furthermore, the random walk on \mathcal{G}^ϵ converges in the scaling limit to Liouville Brownian motion [Ber13, GRV13a] on the limiting LQG surface.*

1.4 Main results

Throughout this subsection we assume we are in the setting of Section 1.3. We make frequent use of the notations in Definitions 1.7 and 1.8. Our first main result gives upper and lower bounds for the scaling dimension of \mathcal{G}^ϵ .

Theorem 1.11. *Let $\gamma \in (0, 2)$ and let*

$$d_- := \frac{2\gamma^2}{4 + \gamma^2 - \sqrt{16 + \gamma^4}} \quad \text{and} \quad d_+ := 2 + \frac{\gamma^2}{2} + \sqrt{2}\gamma. \quad (1.6)$$

Then for each $u > 0$, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}[n^{d_- - u} \leq \#\mathcal{B}_n(0; \mathcal{G}^1) \leq n^{d_+ + u}] = 1$$

where here $\mathcal{B}_n(\cdot)$ is as the graph distance ball of radius n (Definition 1.7).

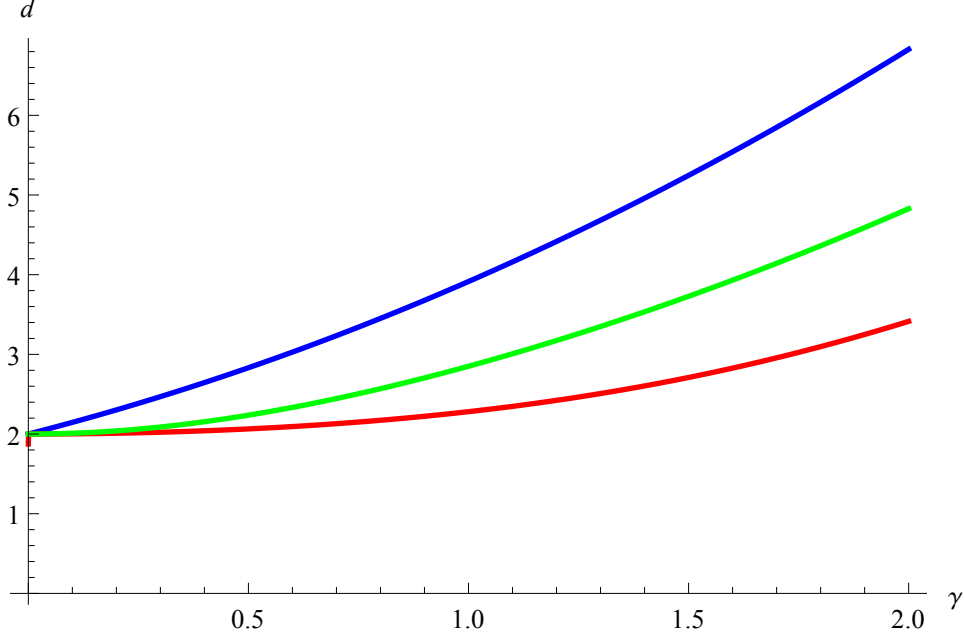


Figure 2: Graphs of the upper and lower bounds d_+ and d_- for the dimension d_γ of γ -LQG (blue and red, respectively) from Theorem 1.11 along with Watabiki prediction (1.2) for d_γ (green), as functions of γ . Produced using Mathematica.

See Figure 2 for a graph of the bounds appearing in Theorem 1.11. By scale invariance, the laws of the graphs \mathcal{G}^1 and \mathcal{G}^ϵ for $\epsilon > 0$ agree, so the statement of Theorem 1.11 is also true with \mathcal{G}^ϵ in place of \mathcal{G}^1 .

By Conjecture 1.10 we expect that the graph distance of \mathcal{G}^ϵ (appropriately re-scaled) is a good approximation for the γ -LQG metric when ϵ is small. Hence it should be the case that in fact $\#\mathcal{B}_n(0; \mathcal{G}^\epsilon) = n^{d_\gamma + o_n(1)}$ with high probability, where d_γ is the dimension of γ -LQG. Therefore Theorem 1.11 can be interpreted as giving upper and lower bounds for d_γ , namely $d_- \leq d_\gamma \leq d_+$. These bounds are consistent with the Watabiki prediction (1.2).

Our next main result is the existence of a tail exponent for distances in the LQG structure graph.

Theorem 1.12. *For $\gamma \in (0, 2)$, the limit*

$$\chi = \chi(\gamma) := \lim_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\text{diam}(\mathcal{G}^\epsilon|_{(0,1]})]}{\log \epsilon^{-1}} \quad (1.7)$$

exists. Furthermore, if we let

$$\xi_- := \frac{1}{d_-} = \frac{1}{2 + \gamma^2/2 + \sqrt{2}\gamma} \quad (1.8)$$

then χ satisfies

$$\xi_- \vee \left(1 - \frac{2}{\gamma^2}\right) \leq \chi \leq \frac{1}{2}. \quad (1.9)$$

The exponent χ will be proven to exist via a sub-additivity argument. The reason for the quantity $1 - 2/\gamma^2$ appearing in (1.9) is as follows. In the case when $\gamma > \sqrt{2}$, the left and right outer boundaries of $\eta'([0, 1])$ touch each other to form ‘bottlenecks’. These bottlenecks correspond to simultaneous running infima of L and R relative to time 0 (which a.s. do not exist for a non-positively correlated Brownian motion). The dimension of the time set for Z such that this is the case is $1 - 2/\gamma^2$ [Eva85], so we expect that there are typically of order $\epsilon^{-(1-2/\gamma^2)}$ cells $\eta'([x - \epsilon, x])$ for $x \in \epsilon\mathbb{Z}$ which intersect a bottleneck. Any path from ϵ to 1 in $\mathcal{G}^\epsilon|_{(0,1]}$ must pass through each of these cells.

If we allow for paths between vertices of $\mathcal{G}^\epsilon|_{[0,1]}$ which traverse vertices in all of \mathcal{G}^ϵ (rather than just in $\mathcal{G}^\epsilon|_{[0,1]}$) then bottlenecks do not pose a problem. Hence we still expect that (in the notation of Definition 1.7) we typically have

$$\text{diam}(\mathcal{G}^\epsilon|_{[0,1]}; \mathcal{G}^\epsilon) = \epsilon^{-1/d_\gamma + o_\epsilon(1)} \quad (1.10)$$

where d_γ is the dimension of γ -LQG; and that $\chi = 1/d_\gamma$ when there are either few or no bottlenecks. There are no bottlenecks for $\gamma \leq \sqrt{2}$, and according to the Watabiki prediction (1.2), we have $1/d_\gamma \leq 1 - 2/\gamma^2$ when $\gamma \geq \sqrt{8/3}$. This leads to the following conjecture.

Conjecture 1.13. *Let d_γ be the dimension of γ -LQG. Then there exists $\gamma_* \in (\sqrt{2}, \sqrt{8/3}]$ such that for $\gamma \in (0, \gamma_*]$ we have $\chi = 1/d_\gamma$.*

Remark 1.14. We expect that it is possible to prove using the same estimates used to prove the lower bound in Theorem 1.11 plus some additional argument that in fact

$$\chi \leq \frac{1}{d_+} \vee \left(1 - \frac{2}{\gamma^2}\right) = \frac{4 + \gamma^2 - \sqrt{16 + \gamma^4}}{2\gamma^2} \vee \left(1 - \frac{2}{\gamma^2}\right) \quad (1.11)$$

where d_+ is as in (1.6). However, the argument needed to deduce (1.11) from the results in this paper requires some rather technical estimates for SLE and LQG and we do not find it to be particularly illuminating. Furthermore, we expect that if γ_* is as in Conjecture 1.13, then for $\gamma \in (0, \gamma_*]$ the expected distance between generic points in \mathcal{G}^ϵ along paths which are allowed to traverse any vertex of \mathcal{G}^ϵ (not just vertices in $\mathcal{G}^\epsilon|_{[0,1]}$) is of order $\epsilon^{-\chi + o_\epsilon(1)}$. Once this is known, (1.11) for $\gamma \in [0, \gamma_*]$ becomes a trivial consequence of the estimates of this paper. We will say more about what is needed to prove (1.11) in Remark 3.13. See Figure 3 for a graph of our upper and lower bounds for χ .

Our next result estimates the probability that distances between vertices of $\mathcal{G}^\epsilon|_{[0,1]}$ are of order $\epsilon^{-\chi + o_\epsilon(1)}$. We get an upper bound which holds except on an event of probability decaying faster than any power of ϵ and a lower bound which holds on an event of probability decaying slower than any power of ϵ .

Theorem 1.15. *Let $\gamma \in (0, 2)$ and let χ be as in Theorem 1.12. For $u > 0$, we have (using Notation 1.3)*

$$\mathbb{P}[\text{diam}(\mathcal{G}^\epsilon|_{[0,1]}) > \epsilon^{-\chi - u}] \leq o_\epsilon^\infty(\epsilon). \quad (1.12)$$

Furthermore, suppose $s, t \in [0, 1]$ with $s < t$ and let x_s^ϵ (resp. x_t^ϵ) be the element of $(0, 1]_{\epsilon\mathbb{Z}}$ closest to s (resp. t). Then

$$\mathbb{P}[\text{dist}(x_s^\epsilon, x_t^\epsilon; \mathcal{G}^\epsilon|_{[0,1]}) \geq \epsilon^{-\chi + u}] \geq \epsilon^{o_\epsilon(1)}. \quad (1.13)$$

Theorem 1.15 implies in particular that in the setting of that theorem, it holds for each $p > 0$ that

$$\lim_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\text{diam}(\mathcal{G}^\epsilon|_{[0,1]})^p]}{\log \epsilon^{-1}} = \lim_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\text{dist}(x_s^\epsilon, x_t^\epsilon; \mathcal{G}^\epsilon|_{[0,1]})^p]}{\log \epsilon^{-1}} = \chi p. \quad (1.14)$$

Note that Theorem 1.15 does not state that the diameter of $\mathcal{G}^\epsilon|_{[0,1]}$ is at least $\epsilon^{-\chi + o_\epsilon(1)}$ with uniformly positive probability. We expect that this is the case with probability tending to 1 as $\epsilon \rightarrow 0$ for each $\gamma \in (0, 2)$, but we prove this only in the special case when $\gamma = \sqrt{2}$ (i.e. when the coordinates of the peanosphere Brownian motion Z are independent).

Theorem 1.16. *Let $\gamma = \sqrt{2}$ and let χ be as in Theorem 1.12. For $u > 0$, one has*

$$\lim_{\epsilon \rightarrow 0} \mathbb{P} \left[\max_{x \in (0, 1]_{\epsilon\mathbb{Z}}} \text{dist}(x, [1, \infty)_{\epsilon\mathbb{Z}}; \mathcal{G}^\epsilon|_{(0, \infty)}) \geq \epsilon^{-\chi + u} \right] = 1. \quad (1.15)$$

In particular,

$$\lim_{\epsilon \rightarrow 0} \frac{\log \text{diam}(\mathcal{G}^\epsilon|_{[0,1]})}{\log \epsilon^{-1}} = \chi \quad \text{in probability.}$$

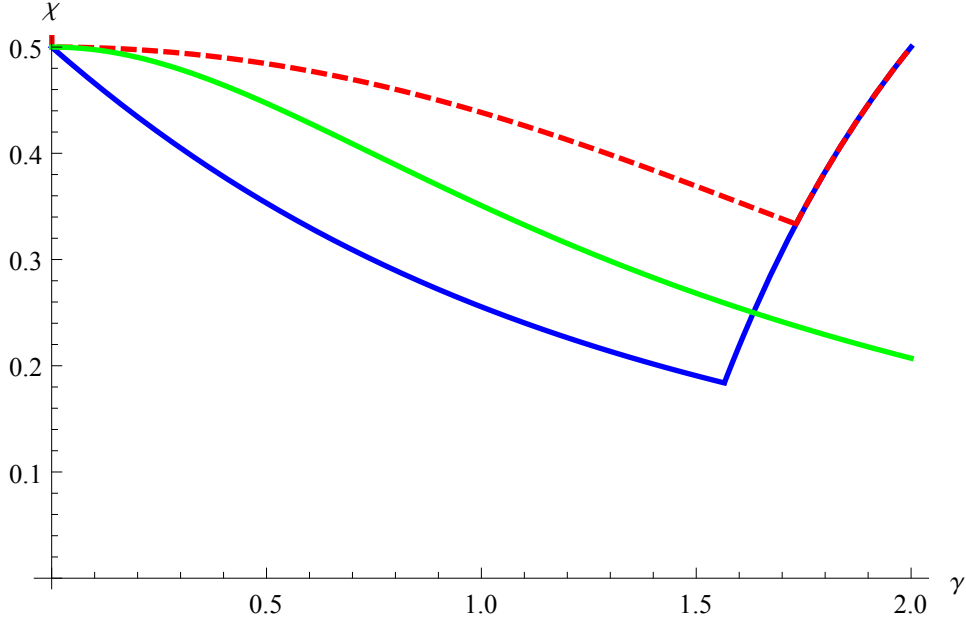


Figure 3: Graphs of the lower and upper bounds for χ (blue and dashed red, respectively) from Theorem 1.12 and Remark 1.14, respectively, along with the reciprocal $1/d_\gamma$ of the dimension of γ -LQG as predicted by Watabiki (green; see (1.2)), as a functions of γ , produced using Mathematica. The reason the red line is dotted is that the upper bound for χ is not proven rigorously; see Remark 1.14. The blue and green curves cross at $(\sqrt{8/3}, 1/4)$ and the red and blue curves meet at $(\sqrt{3}, 1/3)$. The “kink” in the blue curve is located at approximately $(1.56542, 0.183854)$. Conjecture 1.13 states that there is a $\gamma_* \in (\sqrt{2}, \sqrt{8/3})$ such that $\chi = 1/d_\gamma$ for $\gamma \in (0, \gamma_*]$.

The maximum appearing in (1.15) is at most the diameter of $\mathcal{G}^\epsilon|_{(0,1]}$ with respect to the internal graph distance on $\mathcal{G}^\epsilon|_{(0,\infty)}$ (recall Definitions 1.7 and 1.8). We will actually prove a more quantitative version of Theorem 1.16 (see Proposition 8.1 below) which gives an estimate for the rate of convergence (in terms of u and χ) and a lower bound for a slightly smaller quantity. Estimates which give lower bounds for the distance of an interior point of a cell to the boundary of a cell are of particular interest since they may be used to justify that χ is the relevant exponent for the diameter of a cell relative to the whole structure graph \mathcal{G}^ϵ , not only for the diameter relative to $\mathcal{G}^\epsilon|_{(0,1]}$. Observe that (1.15) (or the stronger version in Proposition 8.1) are partial results for establishing estimates of this kind.

The main difference between the estimates in this paper and the sorts of estimates which would be needed to obtain a non-trivial subsequential limit of the rescaled structure graphs $\epsilon^\chi \mathcal{G}^\epsilon|_{(0,1]}$ in the Gromov-Hausdorff topology is the presence of “ $\epsilon^{o_\epsilon(1)}$ ”-type errors. Indeed, if we could replace “ $\epsilon^{-\chi-u}$ ” with “ $C\epsilon^{-\chi}$ ” and “ $o_\epsilon^\infty(\epsilon)$ ” with “ $o_C^\infty(C)$ ” in (1.12) we would have tightness due to the Gromov compactness criterion [BBI01, Theorem 7.4.15]. If we could also replace “ ϵ^u ” with “ C^{-1} ” and “ $\epsilon^{o_\epsilon(1)}$ ” with “ $C^{o_C(1)}$ ” in (1.13), then we would get that any subsequential limit is a non-trivial metric space with positive probability. If we had a version of Theorem 1.16 without an ϵ -dependent error, we would get that any such subsequential limit is non-trivial a.s. in the case when $\gamma = \sqrt{2}$.

1.5 Related works

There are several other recent rigorous results concerning the γ -LQG metric for $\gamma \in (0, 2)$ and its dimension. In this subsection we briefly discuss how these works relate to the present paper.

The papers [MRVZ14, AK14] study the Liouville heat kernel, i.e. the heat kernel of the Liouville Brownian motion [Ber13, GRV13a, GRV13b]. As explained in [MRVZ14], if one assumes a certain relationship between

two exponents which the authors are unable to verify, then the estimates of [MRVZ14] suggest that the dimension d_γ of γ -LQG should satisfy

$$2 + \frac{\gamma^2}{4} \leq d_\gamma \leq \frac{(4 + \gamma^2)^2}{2(2 - \gamma)^2}, \quad (1.16)$$

which is consistent with (1.2). In [AK14], a sharper upper bound is obtained which (subject to the same assumption about relationships between exponents) suggests that $d_\gamma \leq \frac{1}{2}(\gamma + 2)^2$. Our upper and lower bounds from Theorem 1.11 are sharper (closer to the Watabiki prediction) than the upper and lower bounds of [MRVZ14, AK14] for all $\gamma \in (0, 2)$. Note, however, that there are no rigorous mathematical relationships between the exponent bounds of Theorem 1.11 and those of [MRVZ14, AK14]. But we conjecture that the random walk on \mathcal{G}^ϵ converges to Liouville Brownian motion (also see [DMS14, Question 13.2]), which will link the two approaches.

In addition to the structure graph, there is another natural approach to constructing a γ -LQG metric for general values of γ , based on the discrete Gaussian free field. If h is an instance of the discrete GFF on some sub-graph of \mathbb{Z}^2 and $\tilde{\gamma} > 0$, then one obtains a metric on \mathbb{Z}^2 by defining the distance between $z, w \in \mathbb{Z}^2$ to be the infimum of $\sum_{i=1}^n e^{\tilde{\gamma}h(z_i)}$ over all paths z_1, \dots, z_n of vertices in \mathbb{Z}^2 connecting z and w , which is sometimes referred to as *Liouville first passage percolation*. It is natural to expect that the Liouville first passage percolation metric converges in the scaling limit to the γ -LQG metric induced by a continuum GFF for $\tilde{\gamma} = \gamma/d_\gamma$. The relation between $\tilde{\gamma}$ and γ can be explained by observing that for a surface of dimension d_γ , rescaling areas by a constant c should correspond to rescaling lengths by c^{1/d_γ} , and we can obtain such a rescaling by replacing h by $h + \gamma^{-1} \log c$ and $h + (d_\gamma \tilde{\gamma})^{-1} \log c$, respectively.

The recent paper [DD16] proves that for small enough $\tilde{\gamma} > 0$, the Liouville first passage percolation metric corresponding to the discrete GFF on the unit square of side length N , appropriately rescaled, converges along subsequences as $N \rightarrow \infty$ in the Gromov-Hausdorff topology to non-trivial limiting metric spaces which are homeomorphic to the Euclidean unit square. It is also shown in [DZ16] that for small enough $\tilde{\gamma} > 0$, the number of edges of a geodesic for this metric is at least $N^{1+\alpha}$ for a small $\alpha > 0$.

In a sense, the results of [DD16] are orthogonal to the results of the present paper, since the present paper focuses on existence and bounds for scaling exponents and most of the results apply for all values of $\gamma \in (0, 2)$, but we do not prove existence of non-trivial subsequential limiting metrics; whereas [DD16] proves the existence of non-trivial subsequential limits for small values of $\tilde{\gamma}$ but does not explicitly describe the scaling factors.

There are also a number of recent works which focus on estimating exponents for Liouville first passage percolation. The papers [DZ15, DG15] prove some distance estimates for related metrics, defined with h replaced by another log-correlated Gaussian field. The work [DG16b] (c.f. [DG16a]) shows that the expected distance between typical points for the Liouville first passage percolation metric is at most a constant times $N^{1-c \frac{\tilde{\gamma}^{4/3}}{\log \tilde{\gamma}^{-1}}}$ with $c > 0$ a universal constant for small enough $\tilde{\gamma}$; and also proves similar upper bounds for related metrics defined using circle averages or balls of fixed $\tilde{\gamma}$ -LQG mass for a continuum GFF. As pointed out in [DG16b], this estimate appears to be inconsistent with the Watabiki prediction (1.2) since it suggests that $2/d_\gamma \leq 1 - c \frac{\gamma^{4/3}}{\log \gamma^{-1}}$ for small enough γ whereas (1.2) implies that $2/d_\gamma = 1 - O_\gamma(\gamma^2)$.

In contrast to the estimates of [DG16b], our estimates for exponents hold for all $\gamma \in (0, 2)$ and are consistent with the Watabiki prediction. Even though the estimates of [DG16b] are not consistent with the Watabiki prediction, said estimates *are* consistent with the estimates of the present paper. In particular, our lower bound for χ (which we expect to be equal to $1/d_\gamma$ at least for $\gamma \leq \sqrt{2}$) in Theorem 1.12 suggests that $2/d_\gamma \geq 1 - O_\gamma(\gamma)$ as $\gamma \rightarrow 0$, which does not contradict the upper bound $2/d_\gamma \leq 1 - c \frac{\gamma^{4/3}}{\log \gamma^{-1}}$.

As in the case of the Liouville heat kernel estimates, there is presently no rigorous mathematical relationship between the results of this paper and those of [DZ15, DG15, DD16, DG16a, DZ16, DG16b]. The relationship between these two metrics and their connection to the Watabiki prediction (1.2) merits further investigation, especially in light of the aforementioned discrepancy between the estimates of [DG16b] and the Watabiki prediction.

1.6 Discrete intuition

The definition of the LQG structure graphs \mathcal{G}^ϵ is in large part motivated by analogy with discrete models. A number of random planar maps decorated by statistical physics models can be encoded by walks in \mathbb{Z}^2 via discrete analogues of the peanosphere construction. These models include critical-FK planar maps [She16b], bipolar-oriented planar maps [KMSW15], and active planar maps [GKMW16]. For several such models, it has been shown that the corresponding walk in \mathbb{Z}^2 converges in the scaling limit to a correlated two-dimensional Brownian motion whose correlation is the same as that of the Brownian motion in the peanosphere construction for the LQG surface which is expected to be the scaling limit of the random planar map. Such models are said to converge to SLE-decorated LQG in the *peanosphere sense*. Such scaling limit results are proven in [She16b, GMS15, GS15a, GS15b, KMSW15, GHS16, GKMW16]. In the context of FK planar maps, see also [Che15, BLR15] for related results and [GM16a] for a scaling limit result in a stronger topology which is proven using peanosphere scaling limit results.

In the above discrete models, the metric on the random planar map can be described in terms of the walk which encodes the map. Indeed, one can see whether two edges of the map are adjacent from the random walk, and thereby recover the metric by taking the infimum of the lengths of paths consisting of adjacent edges connecting two given points. The condition for two vertices to be adjacent depends on the exact map we are considering but is always some sort of discretization of the condition (1.3) for two cells of the LQG structure graph to be adjacent.

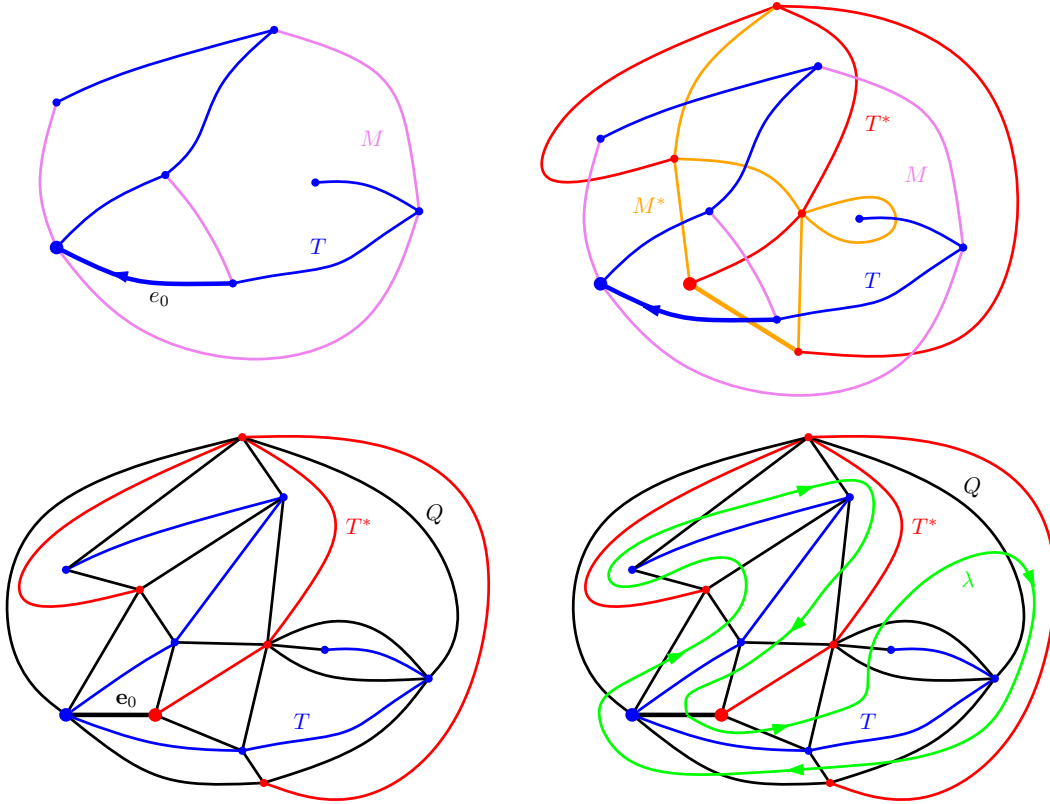


Figure 4: Top left: a planar map M with an oriented root edge e_0 and a spanning tree T . Top right: the dual map M^* and the dual spanning tree T^* . Bottom-left: the quadrangulation Q (whose vertices correspond to the vertices and faces of M) with its root edge e_0 . The graph \mathcal{T} is the adjacency graph on the set of triangles formed by edges of $Q \cup T \cup T^*$. Bottom-right: the space-filling path λ on \mathcal{T} .

We illustrate the above principle for a certain notion of graph distance on maps decorated by spanning trees; see Figure 4. There is a bijection between triples (M, e_0, T) consisting of a planar map with n edges, an oriented root edge e_0 , and a distinguished spanning tree of M ; and the set \mathcal{D}_n of simple walks D on \mathbb{Z}^2

with $2n$ steps which stay in the first quadrant and start and end at $(0, 0)$. This bijection is originally due to Mullin [Mul67], and is explained more explicitly in [Ber07]. See also [She16b, Section 4.1] for another exposition of this bijection as an intermediate step of a more general bijection.

Given such a rooted spanning tree-decorated planar map (M, e_0, T) where M has n edges, one obtains the corresponding walk $D \in \mathcal{D}_n$ as follows. Let M^* be the dual map of M (whose vertices correspond to faces of M) and let T^* be the dual spanning tree of M^* , which consists of edges of M^* which do not cross edges of T . Let Q be the graph whose vertex set is the union of the vertex sets of M and M^* , with two such vertices joined by an edge if and only if they correspond to a face of M and a vertex incident to that face. We define the root edge of Q to be the edge which is adjacent to the terminal endpoint of e_0 and which is the first edge in the counterclockwise direction among all such edges. Then Q is a quadrangulation, and each face of Q is bisected by either an edge of T or an edge of T^* , which divides it into two triangles. Let \mathcal{T} be the dual graph of the set of triangles, i.e. the graph whose vertices are these triangles, with two triangles adjacent if and only if they share an edge.

There is a path λ in \mathcal{T} which snakes between the trees T and T^* , crosses each triangle once, and begins and ends at e_0 (λ is a discrete analogue of the space-filling SLE path η). The path λ crosses each face of Q twice (once for each triangle). For $i \in [1, 2n]_{\mathbb{Z}}$, we declare that $D(i) - D(i-1) = (1, 0)$ (resp. $D(i) - D(i-1) = (0, 1)$) every time that λ crosses a face bisected by an edge of T (resp. T^*) for the first time, and that $D(i) - D(i-1) = (-1, 0)$ (resp. $D(i) - D(i-1) = (0, -1)$) every time λ crosses such a face for the second time. Equivalently, for $i \in [1, 2n]_{\mathbb{Z}}$, the two coordinates of $D(i)$ give the distances from $\lambda(i)$ to the root edge e_0 in the trees T and T^* , respectively. One can also invert this procedure to recover (M, e_0, T) from D .

Each $i \in [1, 2n]_{\mathbb{Z}}$ corresponds to a triangle $\lambda(i)$ in \mathcal{T} , which we think of as a discrete analogue of the cells of the structure graph. If $i_1 < i_2$ and $D = (d, d^*)$, the triangles $\lambda(i_1)$ and $\lambda(i_2)$ share an edge if and only if either

$$d(i_1) \vee d(i_2) < \inf_{j \in [i_1+1, i_2-1]} d^*(j) \quad (1.17)$$

or the same holds with d^* in place of d , where here we take the infimum of the empty set to be ∞ if $i_2 = i_1 + 1$. The condition (1.17) is an exact discrete analogue of (1.5).

If one chooses D uniformly at random from \mathcal{D}_n , then one obtains a uniformly random spanning-tree decorated planar map, which is conjectured to converge in the scaling limit to a $\sqrt{2}$ -LQG sphere decorated by a space-filling SLE₈ curve. Sheffield [She16b] describes a probability measure on \mathcal{D}_n (using the so-called hamburger-cheeseburger model) under which the law of the corresponding map M is that of the uniform measure weighted by the the partition function of the critical Fortuin-Kasteleyn (FK) model for given $q \in (0, 4)$; and the tree T is obtained from a realization of the FK model on M via a deterministic procedure originally due to Bernardi [Ber08]. Such critical FK planar maps are conjectured to converge in the scaling limit to SLE _{κ} -decorated γ -LQG where γ satisfies $q = 2 + 2 \cos(\pi\gamma^2/2)$ and $\kappa = 16/\gamma^2$ [She16b, KN04]. The work [GKMW16] introduces a more general family of probability measures on \mathcal{D}_n (using a generalization of the Sheffield hamburger-cheeseburger model) which encode tree-decorated planar maps corresponding to γ -LQG for all $\gamma \in (0, 2)$.

There are also infinite-volume analogues of the above constructions (alluded to in [She16b] and explained more carefully in [Che15] in the FK case) which encode a Benjamini-Schramm limit [BS01] of spanning tree decorated maps with n edges as $n \rightarrow \infty$. In the infinite-volume case, M , Q , and \mathcal{T} are infinite graphs and $D : \mathbb{Z} \rightarrow \mathbb{Z}^2$ is a bi-infinite random walk. The adjacency condition (1.17) for triangles still holds in this case. The bi-infinite walk D converges in the scaling limit to a correlated two-dimensional Brownian motion with correlation $-\cos(\pi\gamma^2/4)$, where γ is the corresponding LQG parameter (in the finite-volume version D instead converges to a correlated two-dimensional Brownian excursion, as least for FK planar maps [GMS15, GS15a, GS15b]).

Law of large numbers considerations suggest that the graph distances of the maps M and Q and their duals should be well-approximated by deterministic constant multiples of the graph distance on \mathcal{T} at large scales. Alternatively, one can express adjacency conditions for these graphs explicitly in terms of the walk D and observe that they are approximate discrete analogues of (1.5). The above considerations lead naturally to Conjecture 1.10 as well as the following conjecture.

Conjecture 1.17. *Let $\lambda : \mathbb{Z} \rightarrow \mathcal{V}(\mathcal{T})$ be the space-filling path of triangles appearing in the infinite-volume version of the construction above. If the walk D converges in the scaling limit to a correlated Brownian*

motion with correlation $-\cos(\pi\gamma^2/4)$, then with χ as in Theorem 1.12 and $u > 0$, one has

$$\lim_{n \rightarrow \infty} \mathbb{P}[n^{\chi-u} \leq \text{diam}(\lambda([0, n]_{\mathbb{Z}})) \leq n^{\chi+u}] = 1$$

where $\lambda([0, n]_{\mathbb{Z}})$ is given the (internal) graph metric it inherits from \mathcal{T} . The analogous statement holds for other random planar map models which converge to γ -LQG in the peanosphere sense.

1.7 Outline

The remainder of this paper is structured as follows. In Section 2, we will review the definitions of LQG, space-filling SLE, and the peanosphere construction; and prove some basic facts about the structure graphs and about correlated two-dimensional Brownian motion. In Section 3, we will use estimates for SLE-decorated LQG to prove quantitative estimates for distances in the γ -LQG structure graph, which will eventually lead to a proof of Theorem 1.11. These estimates are the only arguments in this paper which require non-trivial facts about LQG and SLE. In Sections 4 and 5, we will prove the existence of the limit in (1.7) for ϵ restricted to powers of 2 using a subadditivity argument. We will also obtain a concentration bound for the diameter of $\mathcal{G}^{2-n}|_{[0,1]}$ for $n \in \mathbb{N}$ in terms of χ . In Section 6, we will prove Theorems 1.12 and 1.15. In Sections 7 and 8, we restrict attention to the case when $\gamma = \sqrt{2}$ and prove Theorem 1.16.

2 Preliminaries

2.1 Background on LQG and SLE

In this subsection we give a brief review of some basic properties of Liouville quantum gravity, space-filling SLE, and the peanosphere construction. Although these objects are the main motivation of this work, their non-trivial properties will only be used explicitly in Section 3. The results in the other sections can be phrased in terms of Brownian motion by means of the peanosphere construction. We refer to the cited references for further background.

2.1.1 Liouville quantum gravity

Fix $\gamma \in (0, 2)$. A γ -Liouville quantum gravity (LQG) surface, as defined in [DS11, She16a, DMS14, MS15c] is an equivalence class of pairs (D, h) where $D \subset \mathbb{C}$ is a simply connected domain and h is a distribution on D (typically some variant of the Gaussian free field on D [She07, SS13, She16a, MS16d, MS13]). Two such pairs (\tilde{D}, \tilde{h}) and (D, h) are declared to be equivalent if there is a conformal map $\phi : \tilde{D} \rightarrow D$ such that

$$\tilde{h} = h \circ \phi + Q \log |\phi'|, \quad \text{for } Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \quad (2.1)$$

As shown in [DS11], an LQG surface comes equipped with a natural volume measure μ_h , which is a limit of regularized versions of $e^{\gamma h(z)} dz$ and is invariant under coordinate changes of the form (2.1). That is, if h and \tilde{h} are related as in (2.1) then a.s. $\mu_h(\phi(A)) = \mu_{\tilde{h}}(A)$ for each Borel set $A \subset \tilde{D}$ [DS11, Proposition 2.1]. Similarly, an LQG surface has a natural length measure ν_h which is defined on certain curves in $D \cup \partial D$ [DS11, Section 6], including ∂D and SLE $_{\kappa}$ -type curves for $\kappa = \gamma^2$ [DS11]. For $k \in \mathbb{N}$, one can define quantum surfaces with k marked points (D, h, x_1, \dots, x_k) for $x_1, \dots, x_k \in D \cup \partial D$ by requiring that the map ϕ in (2.1) preserves the marked points.

In this paper we will mostly be interested in a particular type of LQG surface called an α -quantum cone for $\alpha \in (0, Q)$ (in fact we will almost always take $\alpha = \gamma$). This is an infinite-volume doubly-marked quantum surface $(\mathbb{C}, h, 0, \infty)$ introduced in [DMS14, Section 4.3]. The distribution h is obtained from $h^0 - \alpha \log |\cdot|$, where h^0 is a whole-plane GFF, by “zooming in” near the origin. The distribution h is called the *embedding* of the quantum surface into $(\mathbb{C}, 0, \infty)$ and is not uniquely determined by the equivalence class of $(\mathbb{C}, h, 0, \infty)$. Indeed, by (2.1) one obtains another embedding into $(\mathbb{C}, 0, \infty)$ by replacing h with $h(a \cdot) + Q \log |a|$ for $a \in \mathbb{C}$. There is a natural choice of embedding for a quantum cone called a *circle average embedding*, which is the

embedding used in [DMS14, Definition 4.9] and is defined as follows. For $r > 0$, let $h_r(0)$ be the circle average of h over $\partial B_r(0)$ (as defined in [DS11, Section 3.1]). Then a circle average embedding is one for which

$$\sup\{r > 0 : h_r(0) + Q \log r = 0\} = 1$$

and $h|_{\mathbb{D}}$ agrees in law with $(h^0 - \alpha \log |\cdot|)|_{\mathbb{D}}$, where h^0 is a whole-plane GFF with additive constant chosen so that its circle average over $\partial \mathbb{D}$ is 0. We note that if h is an arbitrary embedding into $(\mathbb{C}, 0, \infty)$ of an α -quantum cone then there exists a random $a \in \mathbb{C}$ such that $h(a) + Q \log |a|$ is a circle average embedding of $(\mathbb{C}, h, 0, \infty)$. The modulus $|a|$ is a deterministic function of h but $\arg a$ is not.

2.1.2 Space-filling SLE

For $\kappa > 4$, a *whole-plane space-filling SLE $_{\kappa}$ from ∞ to ∞* is a variant of SLE $_{\kappa}$ which fills all of \mathbb{C} , even in the case when $\kappa \in (4, 8)$ (so that ordinary SLE $_{\kappa}$ does not fill space). This variant of SLE is defined in [DMS14, Footnote 9], using chordal versions of space-filling SLE constructed in [MS13, Sections 1.2.3 and 4.3]. We will not need many specific facts about space-filling SLE in this paper, since we will primarily study the structure graphs from the Brownian motion (i.e., peanosphere) perspective. So, we only give a brief description of this object here and refer the reader to the above cited papers for more details.

In the case when $\kappa \geq 8$, whole-plane space-filling SLE $_{\kappa}$ is just a two-sided version of chordal SLE $_{\kappa}$. In the case when $\kappa \in (4, 8)$, whole-plane space-filling SLE $_{\kappa}$ is obtained by iteratively filling in the “bubbles” disconnected from ∞ by a two-sided variant of chordal SLE $_{\kappa}$ with SLE $_{\kappa}$ -type curves. If $z \in \mathbb{C}$ and τ_z is the (a.s. unique) time that η hits z , then the interface between $\eta((-\infty, \tau_z])$ and $\eta([\tau_z, \infty))$ is the union of two coupled whole-plane SLE $_{\kappa}(2 - \kappa)$ curves from z to ∞ ([MS13, Theorem 1.1] and [DMS14, Footnote 9]) which intersect each other at points different from z if and only if $\kappa \in (4, 8)$. These curves comprise the left and right outer boundaries of $\eta((-\infty, \tau_z])$.

2.1.3 Peanosphere construction

Let $\gamma \in (0, 2)$ and let $\kappa = 16/\gamma^2$. Let $(\mathbb{C}, h, 0, \infty)$ be a γ -quantum cone and let η be a whole-plane space-filling SLE $_{\kappa}$ independent from h . Suppose we parametrize η by γ -quantum area with respect to h , so that $\eta(0) = 0$ and $\mu_h(\eta([s, t])) = t - s$ for each $s, t \in \mathbb{R}$ with $s < t$. For $t > 0$, let L_t (resp. R_t) be the net change in the quantum length (with respect to h) of the left (resp. right) outer boundary of η relative to time 0. In other words, L_t is the quantum length of the set of points in the left outer boundary of $\eta((-\infty, t])$ which do not belong to the left outer boundary of $\eta((-\infty, 0])$ minus the quantum length of the set of points in the left outer boundary of $\eta((-\infty, 0])$ which do not belong to the left outer boundary of $\eta((-\infty, t])$, and similarly for R_t . Then by [DMS14, Theorem 1.13] there is a deterministic constant $\alpha > 0$ depending only on γ such that $Z_t := (L_t, R_t)$ evolves as a correlated two-sided two-dimensional Brownian motion with

$$\text{Var } L_t = \text{Var } R_t = \alpha |t| \quad \text{and} \quad \text{Cov}(L_t, R_t) = -\alpha \cos\left(\frac{\pi\gamma^2}{4}\right) |t| \quad \forall t \in \mathbb{R}. \quad (2.2)$$

It is shown in [DMS14, Theorem 1.14] that Z a.s. determines (h, η) , modulo rotation, but not in an explicit way. However, one can explicitly describe many functionals of (h, η) in terms of Z . For example, η hits the left (resp. right) outer boundary of $\eta((-\infty, 0])$ and subsequently covers up a boundary arc of non-zero quantum length at time $t > 0$ if and only if t is a running infimum for L (resp. R) relative to time 0. Furthermore, if $t > 0$ then the left and right outer boundaries of $\eta((-\infty, 0])$ intersect at $\eta(t)$ if and only if t is a simultaneous running infimum for L and R relative to time 0. Such simultaneous running infima occur if and only if Z is positively correlated [Shi85] which corresponds precisely to the case when $\kappa \in (4, 8)$.

A *peanosphere* is a random pair (M, η) consisting of a topological space M and a *parametrized* space-filling curve on M , constructed from a correlated two-dimensional Brownian motion in the manner described in Figure 5. It follows from the above discussion that a γ -quantum cone decorated by a whole-plane space-filling SLE $_{\kappa}$ is a canonical embedding of an infinite-volume peanosphere into \mathbb{C} .

2.2 Basic properties of the structure graph

Throughout this subsection, we fix $\gamma \in (0, 2)$ and use the notation of Section 1.3, so in particular $(\mathbb{C}, h, 0, \infty)$ is a γ -quantum cone, η is a whole-plane space-filling SLE $_{\kappa}$ for $\kappa = 16/\gamma^2$ independent from γ and parametrized

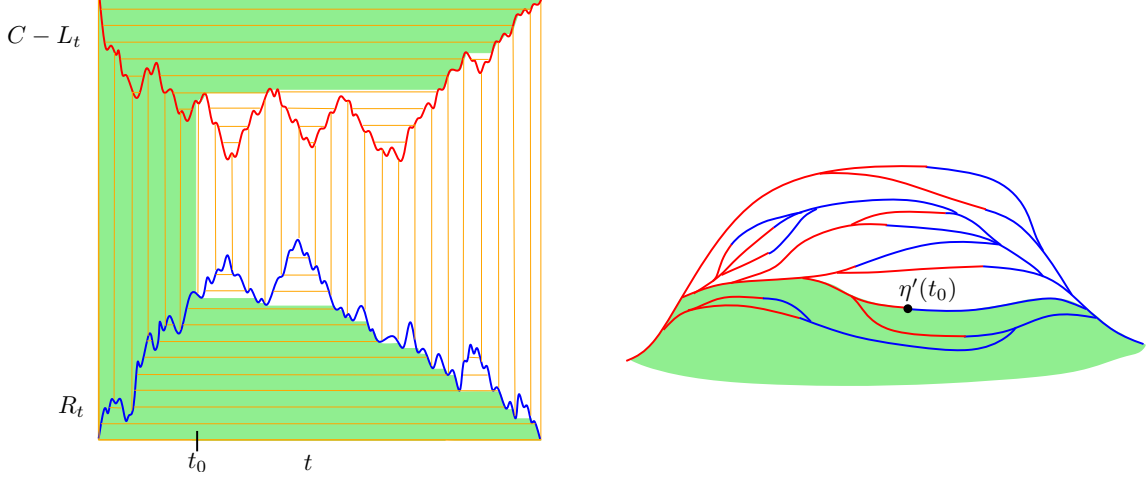


Figure 5: The peanosphere construction of [DMS14] shows how to obtain a topological sphere by gluing together two correlated Brownian excursions $L, R: [0, 1] \rightarrow [0, \infty)$ (a similar construction works when L, R are two-sided Brownian motions, see [DMS14, Footnote 4]). We draw horizontal lines which lie entirely above the graph of $C - L$ or entirely below the graph of R , in addition to vertical lines between the two graphs. We choose $C > 0$ so large that the graphs of $C - L$ and R do not intersect. We then define an equivalence relation by identifying points which lie on the same horizontal or vertical line segment. As explained in [DMS14], it is possible to check using Moore’s theorem [Moo28] that the resulting object is a topological sphere decorated with a space-filling path η where $\eta(t)$ for $t \in [0, 1]$ is the equivalence class of (t, R_t) . The pushforward of Lebesgue measure on $[0, 1]$ under η induces a so-called good measure μ on the sphere (i.e., a non-atomic measure which assigns positive mass to each open set) and η is parameterized according to μ -area, i.e., $\mu(\eta([s, t])) = t - s$ for all $0 \leq s < t \leq 1$. In [DMS14], the resulting structure is referred to as a *peanosphere* because the space-filling path η is the peano curve between the continuum random trees [Ald91a, Ald91b, Ald93] encoded by L and R . As explained in Section 2.1.3, a γ -quantum cone decorated by an independent whole-plane space-filling SLE $_{\kappa}$ parametrized by quantum mass is an embedding of an infinite-volume peanosphere into \mathbb{C} . A finite-volume analogue of this statement appears as [MS15c, Theorem 1.1]. This figure together with a similar caption also appears in [GHM15].

by μ_h -length, and $Z = (L, R)$ is the correlated Brownian motion from Section 2.1.3. We also let

$$\mathcal{F}_t := \sigma(Z_s : s \leq t), \quad \forall t \in \mathbb{R} \quad (2.3)$$

be the filtration generated by the peanosphere Brownian motion.

2.2.1 Boundary lengths

In this subsection we will introduce some additional notation to help us describe the LQG structure graphs. For $a, b \in \mathbb{R}$ with $a < b$, the cell $\eta([a, b])$ has four natural marked boundary arcs, corresponding to the set of points in $\eta([a, b])$ which lie on either the left or right outer boundary of either $\eta((-\infty, b])$ or $\eta'([a, \infty))$. We call these boundary arcs the lower left, lower right, upper left, and upper right boundary arcs. In terms of the peanosphere Brownian motion $Z = (L, R)$, the lower left (resp. right) boundary arc of $\eta([a, b])$ is the image under η of the set of $t \in [a, b]$ such that L (resp. R) attains a running infimum at time t when running forward from time a . Similarly, the upper left (resp. right) boundary arc of $\eta([a, b])$ is the image under η of the set of $t \in [a, b]$ such that L (resp. R) attains a running infimum at time t when running backward from time b .

We will have occasion to consider four marked subsets of the vertex set of $\mathcal{G}^e|_{[0, T]}$ which correspond to the four marked boundary arcs of $\eta([0, T])$ discussed above. See Figure 6 for an illustration.

Definition 2.1. For $\epsilon > 0$ and an (open, closed, or half-open) interval $I \subset \mathbb{R}$, we define the *lower left boundary* of $\mathcal{G}^\epsilon|_I$ to be the set $\underline{\partial}_\epsilon^L I$ of $x \in \epsilon\mathbb{Z} \cap I$ such that the following is true. There is a $y \in \epsilon\mathbb{Z} \setminus I$ with $y < x$ such that the left boundaries of $\eta([x - \epsilon, x])$ and $\eta([y - \epsilon, y])$ share a non-trivial boundary arc. We define the *lower right boundary* $\underline{\partial}_\epsilon^R I$ in the same manner with “right” in place of “left”. We define the *upper left* and *upper right* boundaries $\bar{\partial}_\epsilon^L I$ and $\bar{\partial}_\epsilon^R I$ similarly but with “ $y > x$ ” in place of “ $y < x$ ”. We define

$$\underline{\partial}_\epsilon I := \underline{\partial}_\epsilon^L I \cup \underline{\partial}_\epsilon^R I, \quad \bar{\partial}_\epsilon I := \bar{\partial}_\epsilon^L I \cup \bar{\partial}_\epsilon^R I, \quad \partial_\epsilon I := \underline{\partial}_\epsilon I \cup \bar{\partial}_\epsilon I,$$

so that $\partial_\epsilon I$ is the set of all $x \in I \cap \epsilon\mathbb{Z}$ such that x is adjacent to some element of $\epsilon\mathbb{Z} \setminus I$ in $\mathcal{G}^\epsilon|_I$.

By (1.4), if $x \in \epsilon\mathbb{Z} \cap I$ then $x \in \underline{\partial}_\epsilon^L I$ (resp. $x \in \bar{\partial}_\epsilon^L I$) if and only if the Brownian motion L (resp. its time reversal) attains a running infimum relative to the left (resp. right) endpoint of I at time x . The same holds with “ R ” in place of “ L ”.

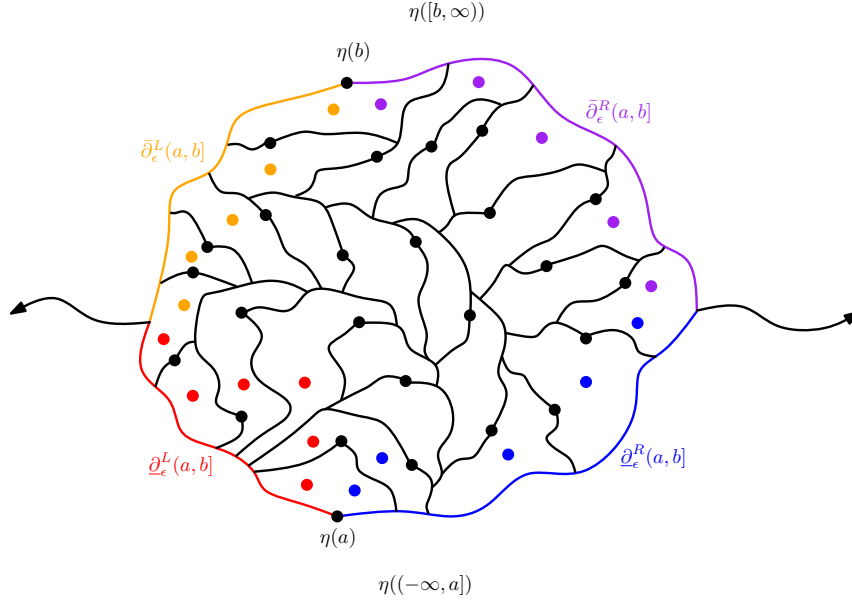


Figure 6: The set $\eta([a, b])$ for $a, b \in \mathbb{R}$ with $a < b$, divided into cells of quantum mass ϵ to obtain the structure graph $\mathcal{G}^\epsilon|_{[a, b]}$. The lower left, lower right, upper left, and upper right boundary arcs of $\eta([a, b])$ are shown in red, blue, orange, and purple, respectively. The cells corresponding to vertices in $\underline{\partial}_\epsilon^L(a, b]$, $\underline{\partial}_\epsilon^R(a, b]$, $\bar{\partial}_\epsilon^L(a, b]$, and $\bar{\partial}_\epsilon^R(a, b]$ are indicated by red, blue, orange, and purple dots, respectively. Note that the first and last cells filled in by η in the time interval $(a, b]$ and the cells which η is filling in when each of L and R attains its infimum on $(a, b]$ each belong to at least two of the four boundary sets. There can, however, be other cells which intersect multiple boundary arcs (there is one such cell in the figure). The set $\underline{\partial}_\epsilon(a, b] = \underline{\partial}_\epsilon^L(a, b] \cup \underline{\partial}_\epsilon^R(a, b]$ is the set of those vertices of $\mathcal{G}^\epsilon|_{[a, b]}$ which are adjacent to vertices of $\mathcal{G}^\epsilon|_{(-\infty, a]}$ and $\bar{\partial}_\epsilon(a, b] = \bar{\partial}_\epsilon^L(a, b] \cup \bar{\partial}_\epsilon^R(a, b]$ is the set of those vertices of $\mathcal{G}^\epsilon|_{[0, T]}$ which are adjacent to vertices of $\mathcal{G}^\epsilon|_{[b, \infty)}$. The quantum boundary lengths of the four marked boundary arcs of $\eta(a, b]$ are given by the four coordinates of the boundary length vector $\Delta_{[a, b]}^Z$.

The following definition is a convenient way of encoding the quantum lengths of the four marked boundary arcs of $\eta([a, b])$.

Definition 2.2. For an interval $I = [a, b] \subset \mathbb{R}$ and a path $X : I \rightarrow \mathbb{R}$, we define the *initial displacement* and the *final displacement* of X over I by

$$\underline{\Delta}_I^X := X_a - \inf_{s \in I} X_s \quad \text{and} \quad \bar{\Delta}_I^X := X_b - \inf_{s \in I} X_s.$$

For the peanosphere Brownian motion $Z = (L, R)$, we define the *boundary length vector* of Z over I by

$$\Delta_I^Z := \left(\underline{\Delta}_I^L, \overline{\Delta}_I^L; \underline{\Delta}_I^R, \overline{\Delta}_I^R \right) \in [0, \infty)^4.$$

The reason for the notation $\underline{\Delta}_I^L$, etc., in Definition 2.2 is that these quantities give the quantum lengths of the four marked boundary arcs of the cell $\eta(I)$ introduced above (this is immediate from the definition of Z ; see Section 2.1.3). One of the main reason for our interest in the objects of Definition 2.2 is the following lemma.

Lemma 2.3. *Let $I \subset \mathbb{R}$ be an interval (possibly infinite or all of \mathbb{R}) and $\epsilon > 0$. The graph $\mathcal{G}^\epsilon|_I$ is measurable with respect to the σ -algebra generated by the boundary length vectors*

$$\left\{ \Delta_{[x-\epsilon, x]}^Z : x \in \epsilon\mathbb{Z} \cap I \right\}, \quad (2.4)$$

notation as in Definition 2.2.

Proof. Let \mathcal{H} be the σ -algebra generated by the set (2.4). We first observe that for each $x_1, x_2 \in \epsilon\mathbb{Z} \cap I$ with $x_1 < x_2$, we have

$$\underline{\Delta}_{[x_1, x_2]}^L = - \inf_{y \in (x_1, x_2] \cap \epsilon\mathbb{Z}} \left(-\underline{\Delta}_{[y-\epsilon, y]}^L + \sum_{z \in [x_1+\epsilon, y-\epsilon] \cap \epsilon\mathbb{Z}} \left(\overline{\Delta}_{[z-\epsilon, z]}^L - \underline{\Delta}_{[z-\epsilon, z]}^L \right) \right)$$

Consequently, $\underline{\Delta}_{[x_1, x_2]}^L \in \mathcal{H}$. Similarly, $\overline{\Delta}_{[x_1, x_2]}^L$, $\underline{\Delta}_{[x_1, x_2]}^R$, and $\overline{\Delta}_{[x_1, x_2]}^R$ are all \mathcal{H} -measurable. On the other hand, the condition (1.5) for $x_1, x_2 \in \epsilon\mathbb{Z} \cap I$ is equivalent to the condition that either

$$\underline{\Delta}_{[x_2-\epsilon, x_2]}^L > \overline{\Delta}_{[x_1, x_2-\epsilon]}^L \quad \text{and} \quad \underline{\Delta}_{[x_1, x_2-\epsilon]}^L < \overline{\Delta}_{[x_1-\epsilon, x_1]}^L \quad (2.5)$$

or the same holds with R in place of L . Thus the event that x_1 and x_2 are adjacent in \mathcal{G}^ϵ is \mathcal{H} -measurable, and we conclude. \square

2.2.2 Comparison of distances

In this subsection we record some elementary observations which allow us to compare distances in the graphs $\mathcal{G}^\epsilon|_{(0, T]}$ for different values of ϵ and T .

Lemma 2.4. *Suppose $n \in \mathbb{N}$, $x_0, x_1 \in (0, 1]_{2^{-n}\mathbb{Z}}$, $y_0 \in \{x_0 - 2^{-n-1}, x_0\}$, and $y_1 \in \{x_1 - 2^{-n-1}, x_1\}$. Then*

$$\text{dist}\left(x_0, x_1; \mathcal{G}^{2^{-n}}|_{(0, 1]}\right) \leq \text{dist}\left(y_0, y_1; \mathcal{G}^{2^{-n-1}}|_{(0, 1]}\right) \leq 2 \text{dist}\left(x_0, x_1; \mathcal{G}^{2^{-n}}|_{(0, 1]}\right). \quad (2.6)$$

In particular, $\text{diam}(\mathcal{G}^{2^{-n-1}}|_{(0, 1]})$ stochastically dominates $\text{diam}(\mathcal{G}^{2^{-n}}|_{(0, 1]})$ and for any $s, t \in [0, 1]$ with $s < t$, $\text{dist}(2^{-n-1}\lceil 2^{n+1}s \rceil, 2^{-n-1}\lfloor 2^{n+1}t \rfloor; \mathcal{G}^{2^{-n-1}}|_{(0, 1]})$ stochastically dominates $\text{dist}(2^{-n}\lceil 2^{n+1}s \rceil, 2^{-n}\lfloor 2^{n+1}t \rfloor; \mathcal{G}^{2^{-n}}|_{(0, 1]})$.

Proof. Suppose $P : [1, |P|]_{\mathbb{Z}} \rightarrow (0, 1]_{2^{-n-1}\mathbb{Z}}$ is a path in $\mathcal{G}^{2^{-n-1}}|_{(0, 1]}$ with $P(1) = y_0$ and $P(|P|) = y_1$. For $i \in [1, |P|]_{\mathbb{Z}}$, let $P'(i) \in (0, 1]_{2^{-n}\mathbb{Z}}$ be defined so that $P(i) \in \{P'(i) - 2^{-n-1}, P'(i)\}$. By definition of $\mathcal{G}^{2^{-n}}$, it follows that P' is a path in $\mathcal{G}^{2^{-n}}|_{(0, 1]}$ of length at most $|P|$ from x_0 to x_1 , which gives the first inequality in (2.6).

For the second inequality, suppose $P : [1, |P|]_{\mathbb{Z}} \rightarrow (0, 1]_{2^{-n}\mathbb{Z}}$ is a path in $\mathcal{G}^{2^{-n}}|_{(0, 1]}$ with $P(1) = x_0$ and $P(|P|) = x_1$. Let $P'(1) = y_0$ and let $P'(2) \in \{x_0 - 2^{-n-1}, x_0\}$ be chosen so that $\eta([P'(2) - 2^{-n-1}, P'(2)])$ shares a non-trivial boundary arc with $\eta([P(2) - 2^{-n}, P(2)])$. For $i \in [2, |P|]_{\mathbb{Z}}$, inductively let $P'(2i-1) \in \{P(i) - 2^{-n-1}, P(i)\}$ be chosen so that $\eta([P'(2i-1) - 2^{-n-1}, P'(2i-1)])$ shares a non-trivial boundary arc with $\eta([P'(2i-2) - 2^{-n-1}, P'(2i-2)])$ and let $P'(2i) \in \{P(i) - 2^{-n-1}, P(i)\}$ be chosen so that $\eta([P'(2i) - 2^{-n-1}, P'(2i)])$ shares a non-trivial boundary arc with $\eta([P(i+1) - 2^{-n}, P(i+1)])$ (unless $i = |P|$, in which case we take $P'(2i) = y_1$). Then P' is a path in $\mathcal{G}^{2^{-n-1}}|_{(0, 1]}$ from y_0 to y_1 with length $2|P|$, and we obtain the second inequality in (2.6). \square

Lemma 2.4 allows us to compare the expected diameters of graphs of the form $\mathcal{G}^\epsilon|_{(0,T]}$ whenever the number of vertices T/ϵ is a non-negative integer power of 2. Our next lemma allows us to extend a weaker form of this comparison to the case when T/ϵ is not a power of 2.

Lemma 2.5. *Suppose $\epsilon > 0$ and $T > \epsilon$. Let $m := \lfloor \log_2(T/\epsilon) \rfloor$ and write $\lfloor T/\epsilon \rfloor = \sum_{j=1}^k 2^{n_j}$ where $n_1, \dots, n_k \in [0, m]_{\mathbb{Z}}$ with $n_1 < \dots < n_k$. Then*

$$\begin{aligned} \mathbb{E}[\text{diam}(\mathcal{G}^\epsilon|_{(0,T]})] &\leq \sum_{j=1}^k \mathbb{E}[\text{diam}(\mathcal{G}^{2^{-n_j}}|_{(0,1]})] \\ &\leq m \mathbb{E}[\text{diam}(\mathcal{G}^{2^{-m}}|_{(0,1]})]. \end{aligned} \quad (2.7)$$

Furthermore,

$$\begin{aligned} \mathbb{E}[\text{dist}(\epsilon, \epsilon \lfloor T/\epsilon \rfloor; \mathcal{G}^\epsilon|_{(0,T]})] &\leq \sum_{j=1}^k \mathbb{E}[\text{dist}(2^{-n_j}, 1; \mathcal{G}^{2^{-n_j}}|_{(0,1]})] \\ &\leq m \mathbb{E}[\text{dist}(2^{-m}, 1; \mathcal{G}^{2^{-m}}|_{(0,1]})]. \end{aligned} \quad (2.8)$$

Proof. By scaling we can assume without loss of generality that $\epsilon = 1$. With n_1, \dots, n_k as in the statement of the lemma, we can write $(0, T]_{\mathbb{Z}} = \bigsqcup_{j=1}^k I_j$, where I_1, \dots, I_k are disjoint and each I_j is the intersection of \mathbb{Z} with some interval and satisfies $\#I_j = 2^{n_j}$. By translation and scale invariance,

$$\mathbb{E}[\text{diam}(\mathcal{G}^1|_{(0,T]})] \leq \sum_{j=1}^k \mathbb{E}[\text{diam}(\mathcal{G}^1|_{I_j})] = \sum_{j=1}^k \mathbb{E}[\text{diam}(\mathcal{G}^{2^{-n_j}}|_{(0,1]})].$$

This proves the first inequality in (2.7). The second inequality follows from the stochastic domination statement for diameters in Lemma 2.4. The estimate (2.8) is proven similarly. \square

2.3 Brownian motion estimates

In this subsection we record some miscellaneous elementary estimates for Brownian motion which we will need several times in the remainder of this article. Throughout, we fix $\gamma \in (0, 2)$ and we let $Z = (L, R)$ be a correlated two-dimensional Brownian motion with variances α and covariance $-\alpha \cos(\pi\gamma^2/4)$, with $\alpha = \alpha(\gamma)$ as in (2.2). We start with a basic continuity estimate.

Lemma 2.6. *There are constants $a, c_0, c_1 > 0$, depending only on γ , such that the following is true. For $n \in \mathbb{N}$, let*

$$F_n := \left\{ \sup_{s_1, s_2 \in [t_1, t_2]} |Z_{s_1} - Z_{s_2}| \leq n(t_2 - t_1)^{1/2}, \forall t_1, t_2 \in [0, 1] \text{ with } t_2 - t_1 \geq 2^{-an^2} \right\}.$$

Then $\mathbb{P}[F_n^c] \leq c_0 e^{-c_1 n^2}$.

Proof. For $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, let

$$E_{n,m} := \left\{ \sup_{s_1, s_2 \in [(k-1)2^{-m}, k2^{-m}]} |Z_{s_1} - Z_{s_2}| \leq n2^{-m/2-2}, \forall k \in [1, 2^m]_{\mathbb{Z}} \right\}.$$

By the Gaussian tail bound and the union bound,

$$\mathbb{P}[E_{n,m}^c] \leq c'_0 2^m e^{-c'_1 n^2}, \quad \forall n \in \mathbb{N}, \forall m \in \mathbb{N} \cup \{0\}$$

for constants $c'_0, c'_1 > 0$ depending only on γ . Hence if we choose $a < c'_1/8$, then

$$\mathbb{P}\left[\bigcap_{m=0}^{\lceil 8an^2 \rceil} E_{n,m}\right] \geq 1 - c'_0 e^{-c'_1 n^2} \sum_{m=0}^{\lceil 8an^2 \rceil} 2^m \geq 1 - c_0 e^{-c_1 n^2}$$

for universal constants $c_0, c_1 > 0$. Now suppose $\bigcap_{m=0}^{\lceil 8an^2 \rceil} E_{n,m}$ occurs and $t_1, t_2 \in [0, 1]$ with $t_2 - t_1 \geq 2^{-an^2}$. Choose $m \in \mathbb{N}$ such that $2^{-m-1} \leq t_2 - t_1 \leq 2^{-m}$ and note that $m \leq \frac{a}{8}n^2$. We can find $k \in [1, 2^m]_{\mathbb{Z}}$ with $t_2, t_1 \in [(k-1)2^{-m}, (k+1)2^{-m}]$. Then for $s_1, s_2 \in [t_1, t_2]$,

$$|Z_{s_1} - Z_{s_2}| \leq |Z_{s_1} - Z_{k2^{-m}}| + |Z_{s_2} - Z_{k2^{-m}}| \leq n(t_2 - t_1)^{1/2}. \quad (2.9)$$

Therefore F_n occurs. \square

Next, we have an estimate for the probability of an “approximate $\pi/2$ -cone time” for Z , which is proven in [Shi85]. In the statement of the lemma, for a set $A \subset \mathbb{C}$ we define $B_\epsilon(A) \subset \mathbb{C}$ to be the set of points which have Euclidean distance $< \epsilon$ to A .

Lemma 2.7. *Let $T > 0$ and $\delta_L, \delta_R > 0$ and set $\bar{\delta}_L := \delta_L \wedge T^{1/2}$ and $\bar{\delta}_R := \delta_R \wedge T^{1/2}$. Then*

$$\mathbb{P} \left[\inf_{t \in [0, T]} L_t \geq -\delta_L, \inf_{t \in [0, T]} R_t \geq -\delta_R \right] \asymp T^{-2/\gamma^2} (\bar{\delta}_L \wedge \bar{\delta}_R) (\bar{\delta}_L \vee \bar{\delta}_R)^{4/\gamma^2 - 1} \quad (2.10)$$

with implicit constant depending only on γ . Furthermore, suppose that A is the image of a smooth path $[0, 1] \rightarrow [0, \infty)^2$ starting from 0 and ending at $z \in [0, \infty)^2$, and let $\epsilon > 0$. Then

$$\mathbb{P} \left[Z([0, T]) \subset B_\epsilon(A), Z(T) \in B_\epsilon(z) \mid \inf_{t \in [0, T]} L_t \geq -\delta_L, \inf_{t \in [0, T]} R_t \geq -\delta_R \right] \succeq 1 \quad (2.11)$$

with the implicit constant depending on A, ϵ, T , and γ but not δ_L or δ_R .

Proof. The estimate (2.10) follows from [Shi85, Equation (4.3)] (applied with $z = \delta_L + i\delta_R$) after applying a linear transformation to Z to get an uncorrelated Brownian motion (c.f. the proof of [GMS15, Lemma 2.2]). The estimate (2.11) follows from [Shi85, Theorem 2] together with the analogous statements for unconditioned Brownian motion and for Brownian motion conditioned to stay in a cone. \square

We also have an estimate for the cardinality of the boundary of the graph $\mathcal{G}^\epsilon|_{(0, T]}$, as defined in Definition 2.1, which is really just an estimate for Brownian motion.

Lemma 2.8. *For $T > 0$ and $0 < \epsilon < T$, we have (in the notation of Definition 2.1)*

$$\mathbb{E}[\#\partial_\epsilon(0, T)] \asymp T^{1/2} \epsilon^{-1/2}$$

with implicit constant depending only on γ .

Proof. By symmetry, it suffices to show that $\mathbb{E}[\#\bar{\partial}_\epsilon^L(0, T)] \asymp T^{1/2} \epsilon^{-1/2}$. If $x \in (0, T - \epsilon]_{\epsilon\mathbb{Z}}$, then $x \in \bar{\partial}_\epsilon^L(0, T]$ if and only if

$$\inf_{t \in [x, T]} (L_t - L_x) > \inf_{t \in [x - \epsilon, x]} (L_t - L_x). \quad (2.12)$$

The random variables on the left and right sides of (2.12) are independent. By the reflection principle, the random variable on the right has the law of -1 times the modulus of a centered Gaussian random variable with variance $\alpha\epsilon$. For each $r > 0$,

$$\mathbb{P} \left[\inf_{t \in [x, T]} (L_t - L_x) > -r \right] \asymp (T - x)^{-1/2} (r \wedge (T - x)^{1/2}).$$

By combining these observations, we find that

$$\mathbb{P} \left[x \in \bar{\partial}_\epsilon^L(0, T] \right] \asymp (T - x)^{-1/2} \epsilon^{1/2}, \quad \forall x \in (0, T - \epsilon]_{\epsilon\mathbb{Z}}.$$

Clearly, $\mathbb{P}[\epsilon \lfloor T/\epsilon \rfloor \in \bar{\partial}_\epsilon^L(0, T)] = 1$. We conclude by summing over all $x \in (0, T]_{\epsilon\mathbb{Z}}$. \square

3 Quantitative distance bounds

In this section we will use space-filling SLE and LQG to prove estimates which will eventually lead to the bounds in Theorem 1.11 as well as the lower bound for χ in Theorem 1.12. This is the only section of the paper in which we directly use non-trivial facts about SLE and LQG; the rest of our arguments can be formulated solely in terms of Brownian motion.

3.1 Lower bound for the diameter

In this subsection we will prove the following lower bounds for distances in the LQG structure graph, which will eventually lead to the upper bound in Theorem 1.11 and the lower bound for χ in Theorem 1.12.

Proposition 3.1. *Let $\xi_- = (2 + \gamma^2/2 + \sqrt{2}\gamma)^{-1}$ be as in (1.8). With probability tending to 1 as $\epsilon \rightarrow 0$,*

$$\text{dist}(0, \partial_\epsilon(-1, 1]; \mathcal{G}^\epsilon) \geq \epsilon^{-\xi_- + o_\epsilon(1)}, \quad (3.1)$$

with ∂_ϵ as in Definition 2.1; and

$$\text{dist}(\epsilon, 1; \mathcal{G}^\epsilon|_{(0,1]}) \geq \epsilon^{-\xi_- \vee (1-2/\gamma^2) + o_\epsilon(1)}. \quad (3.2)$$

In particular,

$$\mathbb{E}[\text{diam}(\mathcal{G}^\epsilon|_{(0,1]})] \geq \epsilon^{-\xi_- \vee (1-2/\gamma^2) + o_\epsilon(1)}. \quad (3.3)$$

To prove Proposition 3.1, we will first use some basic estimates for LQG to prove an upper bound for the number of space-filling SLE cells $\eta([x - \epsilon, x]_\mathbb{Z})$ with $x \in (-1, 1]_{\epsilon\mathbb{Z}}$ with at least a given *Euclidean* diameter (Lemma 3.3). We know that $\eta([-1, 1])$ a.s. contains a Euclidean ball centered at 0. Our estimates on the Euclidean size of cells will lead to a lower bound for the minimal number of cells in a path in \mathcal{G}^ϵ from 0 to a cell which lies outside this Euclidean ball, which will allow us to conclude Proposition 3.1. We first need the following elementary estimate for the LQG area measure of a γ -quantum cone.

Lemma 3.2. *Let h be a whole-plane GFF, normalized so that its circle average over $\partial B_1(0)$ is 0 or let h be the circle average embedding of a γ -quantum cone in $(\mathbb{C}, 0, \infty)$ (recall Section 2.1.1). Also fix $r \in (0, 1/2)$ and $p \geq 0$. For each $z \in B_{1-r}(0) \setminus B_r(0)$ and each $\epsilon > 0$,*

$$\mathbb{P}\left[\mu_h(B_\epsilon(z)) \leq \epsilon^{2+\gamma^2/2+p}\right] \leq \epsilon^{\frac{p^2}{2\gamma^2} + o_\epsilon(1)}$$

with the implicit constant and the rate of convergence of the $o_\epsilon(1)$ depending only on r .

Proof. If h is a whole-plane GFF on \mathbb{C} , normalized so that its circle average over $\partial B_1(0)$ is 0, then the restriction of $h - \gamma \log |\cdot|$ to $B_1(0)$ agrees in law with the restriction to $B_1(0)$ of the circle average embedding of a γ -quantum cone (see, e.g., the discussion just after [DMS14, Definition 4.9]). Hence it suffices to treat the case where h is a whole-plane GFF. Let $h_\epsilon(\cdot)$ be the circle average process for h . By [GHM15, Lemma 3.8] (c.f. [DS11, Lemma 4.5]), for each $u > 0$ we have

$$\mathbb{P}\left[\mu_h(B_\epsilon(z)) \leq \epsilon^{2+\gamma^2/2+u} e^{\gamma h_\epsilon(z)}\right] = o_\epsilon^\infty(\epsilon).$$

Furthermore, $h_\epsilon(z)$ is a centered Gaussian random variable with variance at most $\log \epsilon^{-1} + O_\epsilon(1)$ [DS11, Section 3.1], so the Gaussian tail bound implies

$$\mathbb{P}\left[h_\epsilon(z) \leq \frac{p-u}{\gamma} \log \epsilon\right] \leq \epsilon^{\frac{(p-u)^2}{2\gamma^2}}.$$

Since u is arbitrary the statement of the lemma follows. \square

Lemma 3.3. *Let h be the circle average embedding of a γ -quantum cone in $(\mathbb{C}, 0, \infty)$. Let η be a space-filling SLE_κ from ∞ to ∞ in \mathbb{C} independent from h . For $\alpha > 0$, let*

$$f(\alpha) := \begin{cases} 2\alpha - \frac{\alpha}{2\gamma^2} \left(\frac{1}{\alpha} - 2 - \frac{\gamma^2}{2} \right)^2, & \alpha \leq \frac{2}{4+\gamma^2} \\ (2\alpha) \wedge 1, & \alpha > \frac{2}{4+\gamma^2} \end{cases} \quad (3.4)$$

Then for $0 < r_1 < r_2 < 1$, it holds with probability tending to 1 as $\epsilon \rightarrow 0$ that the number of $x \in (-1, 1]_{\epsilon\mathbb{Z}}$ such that $\eta([x - \epsilon, x]) \subset B_{r_2}(0) \setminus B_{r_1}(0)$ and $\text{diam } \eta([x - \epsilon, x]) \geq \epsilon^\alpha$ is at most

$$\begin{cases} 0, & \alpha < \frac{2}{(2+\gamma)^2} \\ \epsilon^{-f(\alpha)+o_\epsilon(1)}, & \alpha \geq \frac{2}{(2+\gamma)^2}. \end{cases}$$

Proof. Fix $u \in (0, 1)$. For $x \in (-1, 1]_{\epsilon\mathbb{Z}}$, let δ_x be the radius of the largest Euclidean ball contained in $\eta([x - \epsilon, x])$. Also let z_x be the center of this ball. We claim that with probability tending to 1 as $\epsilon \rightarrow 0$, we have

$$\text{diam } \eta([x - \epsilon, x]) \leq \delta_x^{1-u}, \quad \forall x \in (-1, 1]_{\epsilon\mathbb{Z}} \text{ with } \eta([x - \epsilon, x]) \subset B_1(0). \quad (3.5)$$

To see this, we first observe that η is a.s. continuous when parametrized by quantum mass with respect to h , so we can find a deterministic function $\epsilon \mapsto \rho(\epsilon)$ such that $\lim_{\epsilon \rightarrow 0} \rho(\epsilon) = 0$ and with probability tending to 1 as $\epsilon \rightarrow 0$, we have

$$\sup_{x \in (-1, 1]_{\epsilon\mathbb{Z}}} \text{diam } \eta([x - \epsilon, x]) \leq \rho(\epsilon). \quad (3.6)$$

For $\delta > 0$, let \mathcal{E}_δ be the event that the following is true. For each $\tilde{\delta} \leq \delta$ and each $a, b \in \mathbb{R}$ with $a < b$, $\eta([a, b]) \subset B_1(0)$, and $\text{diam } \eta([a, b]) \geq \tilde{\delta}^{1-u}$, the set $\eta([a, b])$ contains a Euclidean ball of radius at least $\tilde{\delta}$. By [GHM15, Proposition 3.2], we have $\mathbb{P}[\mathcal{E}_\delta] \rightarrow 1$ as $\delta \rightarrow 0$, so with $\rho(\epsilon)$ as in (3.6), we have $\lim_{\epsilon \rightarrow 0} \mathbb{P}[\mathcal{E}_{\rho(\epsilon)}] \rightarrow 1$ as $\epsilon \rightarrow 0$. If $\mathcal{E}_{\rho(\epsilon)}$ occurs and (3.6) holds, then (3.5) holds.

For $\alpha > 0$ let $\mathcal{D}_\alpha^\epsilon$ be the set of $w \in (B_{r_2}(0) \setminus B_{r_1}(0)) \cap \frac{\epsilon^\alpha}{4}\mathbb{Z}^2$ with $\mu_h(B_{\epsilon^\alpha}(w)) \leq \epsilon$. By Lemma 3.2 (applied with ϵ^α in place of ϵ and $p = (1/\alpha - 2 - \gamma^2/2) \vee 0$) and the union bound,

$$\mathbb{E}[\#\mathcal{D}_\alpha^\epsilon] \leq \epsilon^{-f(\alpha)+o_\epsilon(1)}.$$

Here we note that trivially $\#\mathcal{D}_\alpha^\epsilon \leq O_\epsilon(\epsilon^{-2\alpha})$. By the Chebyshev inequality, we find that with probability tending to 1 as $\epsilon \rightarrow 0$,

$$\#\mathcal{D}_\alpha^\epsilon \leq \begin{cases} 0, & \alpha < \frac{2}{(2+\gamma)^2} \\ \epsilon^{-f(\alpha)+o_\epsilon(1)}, & \alpha \in \left[\frac{2}{(2+\gamma)^2}, \frac{2}{4+\gamma^2} \right]. \end{cases} \quad (3.7)$$

Here we note that $f(\alpha) < 0$ for $\alpha \in \left(0, \frac{2}{(2+\gamma)^2}\right)$.

Now suppose that (3.5) holds and (3.7) holds with $\alpha/(1-u)$ in place of α . If $x \in (-1, 1]_{\epsilon\mathbb{Z}}$ with $\text{diam } \eta([x - \epsilon, x]) \geq \epsilon^\alpha$, then by (3.5) we have $\delta_x \geq \epsilon^{\alpha/(1-u)}$. Since $\mu_h(\eta([x - \epsilon, x])) = \epsilon$ by definition, there is a $w \in \mathcal{D}_{\alpha/(1-u)}^\epsilon$ with $B_{\epsilon^\alpha}(w) \subset B_{\delta_x}(z_x)$. By (3.7), the number of such x is 0 if $\alpha/(1-u) < 2/(2+\gamma)^2$ and at most $\epsilon^{-f(\alpha)+o_u(1)+o_\epsilon(1)}$ if $\alpha/(1-u) \geq 2/(2+\gamma)^2$, with the $o_u(1)$ independent of ϵ . Since u is arbitrary and $f\left(\frac{2}{4+\gamma^2}\right) = \frac{4}{2+\gamma^2}$, we conclude. \square

Proof of Proposition 3.1. Given $\delta > 0$, we can choose $r_2 \in (0, 1)$ such that with probability at least $1 - \delta$, we have

$$B_{r_2}(0) \subset \eta'([-1, 1]). \quad (3.8)$$

Henceforth fix such an r_2 , a radius $r_1 \in (0, r_2)$, and a parameter $u > 0$. Let $\frac{2}{(2+\gamma)^2} - u = \alpha_0 < \dots < \alpha_N = \frac{1}{2+\gamma^2/2}$ be a partition of $\left[\frac{2}{(2+\gamma)^2} - u, \frac{1}{2+\gamma^2/2}\right]$ with $\alpha_k - \alpha_{k-1} \leq u$ for each $k \in [1, N]_{\mathbb{Z}}$ and $N \asymp u^{-1}$. For $k \in [1, N]_{\mathbb{Z}}$, let A_k^ϵ be the set of $x \in (-1, 1]_{\epsilon\mathbb{Z}}$ with $\eta([x - \epsilon, x]) \subset B_{r_2}(0) \setminus B_{r_1}(0)$ and $\epsilon^{\alpha_k} \leq \text{diam } \eta([x - \epsilon, x]) <$

ϵ^{α_k-1} . Also let A_0^ϵ be the set of $x \in (-1, 1]_{\epsilon\mathbb{Z}}$ with $\eta([x-\epsilon, x]) \subset B_{r_2}(0) \setminus B_{r_1}(0)$ and $\text{diam } \eta([x-\epsilon, x]) \geq \epsilon^{\alpha_0}$. By Lemma 3.3, it holds with probability tending to 1 as $\epsilon \rightarrow 0$ that for each $k \in [0, N]_{\mathbb{Z}}$,

$$A_0^\epsilon = \emptyset \quad \text{and} \quad \#A_k^\epsilon \leq \epsilon^{-f(\alpha_k)+o_u(1)} \quad \forall k \in [1, N]_{\mathbb{Z}} \quad (3.9)$$

where here $f(\cdot)$ is as in (3.4) and the $o_u(1)$ is independent of ϵ .

Suppose now that (3.8) and (3.9) both hold. The condition (3.9) implies that the total diameter of the cells corresponding to elements of $\bigcup_{j=0}^k A_j^\epsilon$ is at most

$$\sum_{j=1}^k \sum_{x \in A_j^\epsilon} \text{diam } \eta([x-\epsilon, x]) \leq \epsilon^{-f(\alpha_k)+\alpha_k+o_u(1)}, \quad \forall k \in [0, N]_{\mathbb{Z}}. \quad (3.10)$$

Note that the exponent ξ_- in the proposition statement is the unique value for which $-f(\xi_-) + \xi_- = 0$. The quantity (3.10) is smaller than $(r_2 - r_1)/2$ for sufficiently small u and sufficiently small ϵ provided $\alpha_k < \xi_-$. Let P be a path in \mathcal{G}^ϵ from 0 to some $y \in (-1, 1]_{\epsilon\mathbb{Z}}$ with $\eta([y-\epsilon, y]) \subset \mathbb{C} \setminus B_{r_2}(0)$. By (3.8) and (3.10) there exists $n \in \mathbb{N}$ and distinct $y_1, \dots, y_n \in (-1, 1]_{\epsilon\mathbb{Z}}$ each of which is hit by P such that

$$\sum_{i=1}^n \text{diam } \eta([y_i - \epsilon, y_i]) \geq \frac{r_2 - r_1}{2} \quad \text{and} \quad \text{diam } \eta([y_i - \epsilon, y_i]) \leq \epsilon^{\xi_- + o_u(1)} \quad \forall i \in [1, n]_{\mathbb{Z}}.$$

Therefore, $|P| \geq n \geq \epsilon^{-\xi_- + o_u(1) + o_\epsilon(1)}$. Since u and δ are arbitrary we obtain (3.1).

To prove (3.2), let \mathcal{T} be the set of times $t \geq 0$ at which L and R attain a simultaneous running infimum relative to time 0. Let Y^ϵ be the set of $x \in (0, 1]_{\epsilon\mathbb{Z}}$ for which $(x - \epsilon, x] \cap \mathcal{T} \neq \emptyset$. It is easy to see that the Hausdorff dimension of $\mathcal{T} \cap [0, 1]$ has the same law as the same as the Hausdorff dimension of the set of $\pi/2$ -cone times of Z , which by [Eva85, Theorem 1] is a.s. equal to $(1 - 2/\gamma^2) \vee 0$. Consequently, it holds with probability tending to 1 as $\epsilon \rightarrow \infty$ that the number of intervals of length ϵ needed to cover $\mathcal{T} \cap [0, 1]$ is at least $\epsilon^{-(1-2/\gamma^2)+u}$. In particular,

$$\lim_{\epsilon \rightarrow 0} \mathbb{P} \left[\#Y^\epsilon \geq \epsilon^{-(1-2/\gamma^2)+u} \right] = 1.$$

On the other hand, the adjacency condition (1.5) implies that every path from 0 to 1 in $\mathcal{G}^\epsilon|_{(0,1]}$ must pass through every element of Y^ϵ (equivalently, removing an element of Y^ϵ disconnects $\mathcal{G}^\epsilon|_{(0,1]}$ into two pieces). Hence with probability tending to 1 as $\epsilon \rightarrow 0$,

$$\text{dist}(\epsilon, 1; \mathcal{G}^\epsilon|_{(0,1]}) \geq \epsilon^{-(1-2/\gamma^2)+o_\epsilon(1)}.$$

Combining this with (3.1) yields (3.2). \square

3.2 KPZ formula for expected Minkowski dimension

In this subsection we will establish a one-sided KPZ formula from which the lower bound in Theorem 1.11 will be deduced. First we will establish Lemma 3.5, which is a version of the box-counting KPZ formula [DS11, Proposition 1.6], and is proven in a similar manner. Then we will use this lemma along with SLE and GFF estimates to show in Proposition 3.4 that we still get the same KPZ formula if we define the quantum dimension of the set in terms of a cover consisting of space-filling SLE segments with a given quantum measure, instead of dyadic squares. This result is a one-sided Minkowski dimension version of [GHM15, Theorem 1.1] (which concerns Hausdorff dimension instead of Minkowski dimension).

Suppose that h is some variant of the Gaussian free field on \mathbb{C} and that X is a random subset of \mathbb{C} , independent from h , such that $X \subset D$ a.s. for some deterministic and bounded set $D \subset \mathbb{C}$. We will define two different notions of dimension for X . For any $\delta > 0$ let \mathfrak{S}_δ be the set of closed squares with side length δ and endpoints in $\delta\mathbb{Z}^2$. For $z \in \mathbb{C}$ let $S_\delta(z)$ denote the element of \mathfrak{S}_δ which contains z ; $S_\delta(z)$ is uniquely defined except if one or both of the coordinates of z is a multiple of δ , in which case we make an arbitrary choice between the ≤ 4 possibilities when defining $S_\delta(z)$. Let N_δ be the number of such squares which intersect X , i.e.,

$$N_\delta := \#\{S \in \mathfrak{S}_\delta : S \cap X \neq \emptyset\}.$$

The *Euclidean expectation dimension* $\widehat{d}_0 = \widehat{d}_0(X)$ of X for the field h , if it exists, is defined to be the limit

$$\widehat{d}_0 = \lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[N_\delta]}{\log \delta^{-1}} \in [0, 2].$$

Also define for $\epsilon > 0$

$$N^\epsilon := \#\{x \in \epsilon\mathbb{Z} : \eta([x - \epsilon, x]) \cap X \neq \emptyset\}. \quad (3.11)$$

The *quantum expectation dimension* of X for the field h , if it exists, is defined to be the limit

$$2 \lim_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[N^\epsilon]}{\log \epsilon^{-1}} \in [0, 2].$$

The main result of this subsection is the following proposition.

Proposition 3.4. *Assume the Euclidean expectation dimension \widehat{d}_0 of X exists, and that X is independent from h . Let $\widehat{d}_\gamma \in [0, 2]$ be chosen so that*

$$\widehat{d}_0 = \left(1 + \frac{\gamma^2}{4}\right) \widehat{d}_\gamma - \frac{\gamma^2}{8} \widehat{d}_\gamma^2. \quad (3.12)$$

If h is a whole-plane GFF normalized such that the circle average over $\partial\mathbb{D}$ is 0, then

$$2 \limsup_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[N^\epsilon]}{\log \epsilon^{-1}} \leq \widehat{d}_\gamma. \quad (3.13)$$

Furthermore, suppose $\alpha < Q$, that h is the circle average embedding of an α -quantum cone and that D lies at positive distance from 0. There is a constant $c = c(\alpha, \gamma, D) > 0$ such that for each $\epsilon > 0$ and each $u > 0$, we have

$$\mathbb{P}\left[N^\epsilon > \epsilon^{-\widehat{d}_\gamma/2-u}\right] \leq \epsilon^{cu}$$

with the implicit constant independent from ϵ .

Note that the result for the α -quantum cone is weaker than the result for the GFF. The reason for this is that the proof will use a coupling between the GFF and the α -quantum cone for which the associated quantum areas are typically close, but may be very different on subsets of $D \setminus \mathbb{D}$.

We will deduce Proposition 3.4 from a variant of the proposition corresponding to an alternative notion of quantum dimension which is closely related to the box-counting dimension considered in [DS11] but involves squares of side length δ which intersect X and whose δ -neighborhoods have quantum mass at most ϵ , rather than squares which themselves have quantum mass at most ϵ . Let $X \subset D \subset \mathbb{C}$ be a random set as above. Let h be either a whole-plane GFF with additive constant chosen so that the circle average of h over $\partial\mathbb{D}$ is zero, or a zero boundary GFF in a bounded domain $\widetilde{D} \subset \mathbb{C}$ satisfying $\overline{D} \subset \widetilde{D}$. For $S \in \mathfrak{S}_\delta$ the δ -neighborhood \widetilde{S} of S is defined by

$$\widetilde{S} = \{z \in \mathbb{C} : \text{dist}(z, S) < \delta\}. \quad (3.14)$$

We define the *dyadic parent* S_- of S be the unique element of $\mathfrak{S}_{2\delta}$ containing S . For $\epsilon > 0$ we define a (μ_h, ϵ) -box to be a dyadic square $S \in \cup_{k \in \mathbb{Z}} \mathfrak{S}_{2^{-k}}$ which satisfies (in the notation introduced just above) $\mu_h(\widetilde{S}) < \epsilon$ and $\mu_h(\widetilde{S}_-) \geq \epsilon$. In the case of the zero boundary GFF we extend the measure μ_h to a measure on \mathbb{C} by assigning measure 0 to the complement of \widetilde{D} . Let \mathfrak{S}^ϵ be the set of (μ_h, ϵ) -boxes. Since μ_h is non-atomic, for each $z \in \mathbb{C}$ and $\epsilon > 0$ for which none of the coordinates are dyadic, there is a unique square $S^\epsilon(z) \in \mathfrak{S}^\epsilon$ which contains z ; in the case where one or both of the coordinates is dyadic we define $S^\epsilon(z)$ uniquely by also requiring that $S^\epsilon(z) = S_\delta(z)$ for some $\delta = 2^{-k}$, $k \in \mathbb{Z}$, where $S_\delta(z)$ is defined as in the beginning of this section. Note that the difference between our notion of a (μ_h, ϵ) -box, and the notion of a (μ_h, ϵ) -box considered in [DS11, Section 1.4], is that we consider dyadic squares where the *neighborhood* of each square has a certain quantum measure, instead of considering the measure of the squares themselves. See Figure 7 for an illustration.

For $\epsilon > 0$ define $\widehat{N}^\epsilon = \widehat{N}^\epsilon(X)$ to be the number of (μ_h, ϵ) -boxes needed to cover X , i.e.,

$$\widehat{N}^\epsilon = \#\{S \in \mathfrak{S}^\epsilon : S \cap X \neq \emptyset\}.$$

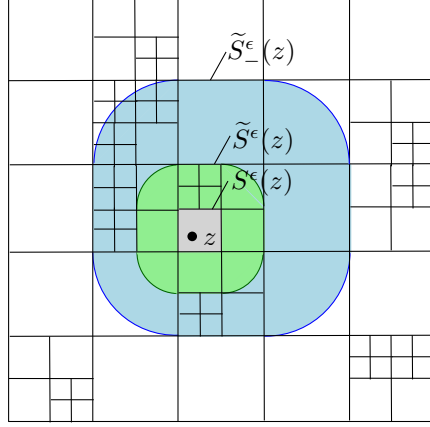


Figure 7: The set of (μ_h, ϵ) -boxes on the figure is the set of squares which do not contain any smaller squares. The figure illustrates various neighborhoods associated with $z \in \mathbb{C}$. The quantum dimension of a random fractal X is defined in terms of the number of squares $S^\epsilon(z)$ needed to cover X . The set $\tilde{S}^\epsilon(z)$ is a neighborhood of $S^\epsilon(z)$, while $\tilde{S}_-^\epsilon(z)$ is a neighborhood of the dyadic parent $S_-^\epsilon(z)$ (which is not labelled on the figure) of $S^\epsilon(z)$. The square $S^\epsilon(z)$ is defined such that $\mu_h(\tilde{S}^\epsilon) < \epsilon$ and $\mu_h(\tilde{S}_-^\epsilon) \geq \epsilon$.

The *box quantum expectation dimension* of X , if it exists, is the limit

$$2 \lim_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\hat{N}^\epsilon]}{\log \epsilon^{-1}} \in [0, 2]. \quad (3.15)$$

The following lemma is a version of [DS11, Proposition 1.6] with our alternative notion of a (μ_h, ϵ) -box. Recall that we assume h is either a whole-plane GFF with unit circle average zero, or a zero boundary GFF.

Lemma 3.5. *If the Euclidean expectation dimension \hat{d}_0 of X exists and X is independent of h , then the box quantum expectation dimension of X exists and is given by \hat{d}_γ , where $\hat{d}_\gamma \in [0, 2]$ solves (3.12).*

Proof. First we consider the case where h is a zero boundary GFF on \tilde{D} . It is sufficient to establish the following two inequalities

$$2 \liminf_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\hat{N}^\epsilon]}{\log \epsilon^{-1}} \geq \hat{d}_\gamma \quad \text{and} \quad 2 \limsup_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\hat{N}^\epsilon]}{\log \epsilon^{-1}} \leq \hat{d}_\gamma. \quad (3.16)$$

The first inequality of (3.16) is immediate, since the number of boxes \hat{N}^ϵ in our cover is at least as large as the number of boxes in the cover considered in [DS11, Proposition 1.6], since each (μ_h, ϵ) -box with our definition is contained in a (μ_h, ϵ) -box with the definition considered in [DS11].

We will now establish the second inequality of (3.16). Let \mathfrak{S}^ϵ_- denote the set of dyadic parents of squares in \mathfrak{S}^ϵ . With \tilde{S} as in (3.14), define the following quantum ϵ -neighborhoods of X :

$$\tilde{S}^\epsilon(X) := \bigcup_{S \in \mathfrak{S}^\epsilon : S \cap X \neq \emptyset} \tilde{S}, \quad \tilde{S}_-^\epsilon(X) := \bigcup_{S \in \mathfrak{S}_-^\epsilon : S \cap X \neq \emptyset} \tilde{S}.$$

For $z \in \mathbb{C}$ define $S_-^\epsilon(z)$ to be the dyadic parent of $S^\epsilon(z)$, and define $\tilde{S}^\epsilon(z)$ (resp. $\tilde{S}_-^\epsilon(z)$) to be the δ -neighborhood of $S^\epsilon(z)$ (resp. the 2δ -neighborhood of $S_-^\epsilon(z)$), where δ is the side length of $S^\epsilon(z)$. The first step of our proof is to reduce the lemma (for the case of a zero boundary GFF) to proving the following estimate:

$$\lim_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\mu_h(\tilde{S}_-^\epsilon(X))]}{\log \epsilon^{-1}} \leq \frac{\hat{d}_\gamma}{2} - 1. \quad (3.17)$$

Let

$$\hat{N}_-^\epsilon = \#\{S \in \mathfrak{S}_-^\epsilon : S \cap X \neq \emptyset\}.$$

By (3.18), which we will explain just below, we see that (3.17) is sufficient to prove the lemma:

$$\lim_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\epsilon \hat{N}_-^\epsilon]}{\log \epsilon^{-1}} = \lim_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\epsilon \hat{N}_-^\epsilon]}{\log \epsilon^{-1}} \leq \lim_{\epsilon \rightarrow 0} \frac{\log \mathbb{E}[\mu_h(\tilde{S}_-^\epsilon(X))]}{\log \epsilon^{-1}}. \quad (3.18)$$

The first equality of (3.18) follows by $\hat{N}_-^\epsilon \leq \hat{N}^\epsilon \leq 4\hat{N}_-^\epsilon$. The second estimate of (3.18) follows since for any $z \in D$ it holds that $\mu_h(\tilde{S}_-^\epsilon(z)) \geq \epsilon$ and $\mu_h(\tilde{S}_-^\epsilon(z) \cap S) > 0$ for at most 9 of the dyadic squares $S \in \mathfrak{S}_-^\epsilon$ which intersect X .

Our justification of (3.17) will be very brief, since a similar argument can be found in [DS11]. Let $\Theta = \mathcal{Z}^{-1}e^{\gamma h} dz dh$ be the rooted probability measure defined in [DS11, Section 3.3]. Proceeding similarly as in [DS11], and letting $\delta = \delta(z, \epsilon)$ denote the (random) side length of $S^\epsilon(z)$ for $(z, h) \sim \Theta$, we see that

$$\lim_{\epsilon \rightarrow 0} \frac{\mathbb{E}[\mu_h(\tilde{S}_-^\epsilon(X))]}{\log \epsilon^{-1}} = \lim_{\epsilon \rightarrow 0} \frac{\mathbb{P}[\tilde{S}_-^\epsilon(z) \cap X \neq \emptyset]}{\log \epsilon^{-1}} = \lim_{\epsilon \rightarrow 0} \frac{\mathbb{E}[\delta^{2-\hat{d}_0}]}{\log \epsilon^{-1}} \leq \frac{\hat{d}_\gamma}{2} - 1. \quad (3.19)$$

In particular, the first equality of (3.19) follows by the argument right after the statement of [DS11, Theorem 4.2], and the second equality of (3.19) follows by the argument of the first paragraph in the proof of [DS11, Theorem 4.2]. The last inequality of (3.19) follows by using that the dyadic squares $S^\epsilon(z)$ have a side length which is smaller than or equal to the corresponding dyadic squares considered in [DS11, Section 1.4], and that the last inequality of (3.19) holds for the dyadic squares considered in [DS11]. The estimate (3.19) implies (3.17), which concludes the proof of the lemma for the case of the zero boundary GFF.

Next we consider the case where h is a whole-plane GFF with additive constant chosen so that its circle average over $\partial\mathbb{D}$ is 0. Choose $R > 0$ such that $D \subset B_{R/4}(0)$. Then $h|_{B_R(0)} = h^0 + \mathfrak{h}$, where h^0 is a zero boundary GFF in $B_R(0)$, and \mathfrak{h} is a harmonic function in $B_R(0)$. By [GHM15, Lemma 3.8], and by using that $\mathfrak{h}(0)$ is the circle average of h around $B_R(0)$, so that $\mathfrak{h}(0)$ is a centered Gaussian random variable with variance $\log(R)$, for any $u > 0$,

$$\mathbb{P}\left[\sup_{z \in B_{R/2}(0)} |\mathfrak{h}(z)| > u \log \delta^{-1}\right] = o_\delta^\infty(\delta).$$

Hence, except on an event of probability $o_\delta^\infty(\delta)$, for any $A \subset B_{R/2}(0)$, we have $\delta^{\gamma u} \mu_{h^0}(A) \leq \mu_h(A) \leq \delta^{-\gamma u} \mu_{h^0}(A)$. Since $u > 0$ is arbitrary, the statement of the lemma for the case of a whole-plane GFF follows from the case of a zero-boundary GFF on $\tilde{D} = B_R(0)$. \square

We will apply the following basic lemma in our proof of Proposition 3.4.

Lemma 3.6. *Let μ_h be the γ -LQG measure associated with a whole-plane GFF with additive constant chosen such that the average around $\partial\mathbb{D}$ is 0. Let $D \subset \mathbb{C}$ be a bounded open set. For $\delta > 0$ let \mathcal{B}_δ be a deterministic collection of at most δ^{-2} Euclidean balls of radius $\delta > 0$ contained in D , and define $A_\delta := \max_{B \in \mathcal{B}_\delta} \mu_h(B)$. Given any $M > 0$ we can find $s = s(M) > 0$ such that $\mathbb{P}(A_\delta > \delta^s) \leq \delta^{sM}$, where the implicit constant is independent of δ .*

Proof. By [GHM15, Lemma 5.2] and the Chebyshev inequality, for any $B \in \mathcal{B}_\delta$, $\beta \in [0, 4/\gamma^2)$ and $s > 0$, we have

$$\mathbb{P}[\mu_h(B) > \delta^s] \leq \delta^{f(\beta) - s\beta + o_\delta(1)},$$

where $f(\beta) = (2 + \frac{\gamma^2}{2})\beta - \frac{\gamma^2}{2}\beta^2$. By the union bound,

$$\mathbb{P}[A_\delta > \delta^s] \leq \delta^{-2} \max_{B \in \mathcal{B}_\delta} \mathbb{P}[\mu_h(B) > \delta^s] \leq \delta^{f(\beta) - 2 - s\beta + o_\delta(1)}.$$

If we choose $\beta \in (1, 4/\gamma^2)$ then for small enough δ and $s > 0$, we have $f(\beta) - 2 - s\beta > sM$, which implies the lemma. \square

Proof of Proposition 3.4. First we consider the case where h is a whole-plane GFF normalized as in the statement of the proposition. Fix a large constant $C > 0$ to be chosen later, depending only on γ . For $\epsilon, u > 0$ let E_ϵ^u be the event that the following is true.

- (i) All squares $S \in \mathfrak{S}^\epsilon$ for which $S \cap D \neq \emptyset$ have Euclidean side length at least ϵ^K , where $K > 0$ is chosen sufficiently large such that the probability of this event is at least $1 - \epsilon^C$. Existence of an appropriate K (independent of ϵ, u) follows from Lemma 3.6 applied with e.g. $\delta = 10\epsilon^K$, $M = 1000$, and \mathcal{B}_δ a collection of balls such that each $S \in \tilde{\mathfrak{S}}_{2\delta}$ for which $S \cap D \neq \emptyset$ is contained in a ball in \mathcal{B}_δ , where $\tilde{\mathfrak{S}}_{2\delta}$ is the set of 2δ -neighborhoods of boxes in $\mathfrak{S}_{2\delta}$.
- (ii) For any interval $I \subset \mathbb{R}$ for which $\delta := \text{diam}(\eta(I)) < \epsilon^u$ and $\eta(I) \cap D \neq \emptyset$ the set $\eta(I) \subset \mathbb{C}$ contains a ball of radius at least δ^{1+u} .

By [GHM15, Proposition 3.2], the probability of the event in (ii) tends to 1 as $\epsilon \rightarrow 0$. In fact, [GHM15, Remark 3.7] together with the results of [HS16] (see in particular Lemma 2.3 and Proposition 6.2 from the latter paper) tells us that the probability of the event in (ii) is of order $1 - o_\epsilon^\infty(\epsilon)$, at a rate depending only on u and the diameter of D . Hence $\mathbb{P}((E_\epsilon^u)^c) \leq \epsilon^C$. If the event 3.2 occurs, then for any dyadic box S of side length $\delta \in (0, \epsilon^K)$, the number of disjoint SLE segments $\eta(I)$ for $I \subset \mathbb{R}$ any interval which intersect both S and $\mathbb{C} \setminus \tilde{S}$ is bounded by δ^{-2u} (c.f. [GHM15, Lemma 5.1]). Therefore the condition (i) implies that on the event E_ϵ^u we have $N^\epsilon \leq \epsilon^{-2Ku} \hat{N}^\epsilon$. Note that $N^\epsilon \leq \mu_h(\tilde{D})\epsilon^{-1}$ for \tilde{D} a slightly larger open set containing \bar{D} , so by Hölder's inequality and the moment estimate in [RV14, Theorem 2.11], we see that $\mathbb{E}[\mathbb{1}_{(E_\epsilon^u)^c} N^\epsilon]$ decays faster than any power of ϵ . It follows that

$$\mathbb{E}[N^\epsilon] \leq \mathbb{E}[\epsilon^{-2Ku} \hat{N}^\epsilon] + \mathbb{E}[\mathbb{1}_{(E_\epsilon^u)^c} N^\epsilon] \leq \epsilon^{-2Ku} \mathbb{E}[\hat{N}^\epsilon].$$

Since $u > 0$ was arbitrary, an application of Lemma 3.5 concludes the proof of the proposition for the case of h a whole-plane GFF.

Now we assume h is the circle average embedding of an α -quantum cone and that D lies at positive distance from 0. Let \tilde{D} be a slightly larger domain containing \bar{D} which also lies at positive distance from 0. By [GHM15, Lemma 3.8], we can couple h with an instance of a whole-plane GFF h^G (normalized as above) satisfying the following property. There is a constant $c = c(\gamma, \alpha) > 0$ such that for each $u \geq 0$, it holds except on an event of probability $\leq \epsilon^{cu}$ that $\epsilon^{u/3} \mu_h(A) \leq \mu_{h^G}(A) \leq \epsilon^{-u/3} \mu_h(A)$ for each $A \subset \tilde{D}$. Let N^ϵ (resp. N_G^ϵ) denote the number of boxes in (3.11) when the field is h (resp. h^G). By the coupling between h and h^G , except on an event of probability ϵ^{cu} , we have $N^\epsilon \leq 10N_G^{\epsilon^{1+u/3}}$. We conclude the proof of the proposition by using the above result for the whole-plane GFF (and we decrease c in the very last step if necessary):

$$\mathbb{P}[N^\epsilon > \epsilon^{-u-d_\gamma}] \leq \mathbb{P}[N^\epsilon > 10N_G^{\epsilon^{1+u/3}}] + \mathbb{P}[10N_G^{\epsilon^{1+u/3}} > \epsilon^{-u-d_\gamma}] \leq \epsilon^{cu}. \quad \square$$

3.3 Lower bound for the cardinality of a ball

In this subsection we will prove the following estimate, which is a more quantitative version of the lower bound in Theorem 1.11.

Proposition 3.7. *Let d_- be as in (1.6). There exists $p = p(\gamma) > 0$ such that for each $u \in (0, 1)$, each $\epsilon > 0$, and each $n \in \mathbb{N}$, we have*

$$\mathbb{P}[\#\mathcal{B}_n(0; \mathcal{G}^\epsilon) \geq n^{d_- - u}] \geq 1 - O_n(n^{-pu^2}). \quad (3.20)$$

Note that by scale invariance, the left side of (3.20) does not depend on ϵ .

Throughout this subsection we assume that $(\mathbb{C}, h, 0, \infty)$ is a γ -quantum cone and that η is a whole-plane space-filling SLE $_\kappa$ independent from h , parametrized by γ -quantum mass with respect to h , as in Section 1.3.

We note that $\hat{d}_\gamma = 1/d_-$ is the solution to the KPZ equation of Proposition 3.4 when the Euclidean dimension \hat{d}_γ is equal to 1. Hence if $z, w \in \mathbb{C}$ are random points at positive distance from 0 which are chosen in a manner which does not depend on h and X is a smooth path from z to w , then Proposition 3.4 implies an upper bound for the number of cells in \mathcal{G}^ϵ needed to cover X , and hence an upper bound for the distance in \mathcal{G}^ϵ between the cells containing z and w . However, we cannot apply this statement directly with $(0, \eta(x))$

in place of (z, w) since h has a γ -log singularity at 0 and $\eta(x)$ is not sampled independently from h (because η is parametrized by quantum mass with respect to h). The γ -log singularity is not a serious issue, and can be overcome by a multi-scale argument (Lemma 3.8). Getting around the fact that $\eta(x)$ is not independent from h , however, will require a bit more work.

We will use Proposition 3.4 to show that for each fixed $x \in (0, \epsilon^u]_{\epsilon\mathbb{Z}}$, it holds with high probability that there exist radii $r_1, r_2 > 0$ such that $\eta(x) \in B_{r_1}(0)$, $0 \in B_{r_2}(\eta(x))$, and each $y \in \mathbb{Z}$ whose corresponding cell $\eta([y - \epsilon, y])$ intersects $\partial B_{r_1}(0)$ (resp. $\partial B_{r_2}(\eta(x))$) lies at \mathcal{G}^ϵ -distance at most $\epsilon^{-1/d_- - u}$ from 0 (resp. x). By concatenating paths, it follows that if this is the case, then x lies at \mathcal{G}^ϵ -graph distance at most $2\epsilon^{-1/d_- - u}$ from 0. Hence with high probability most elements of $(0, \epsilon^u]_{\epsilon\mathbb{Z}}$ lie at \mathcal{G}^ϵ -graph distance at most $2\epsilon^{-1/d_- - u}$ from 0. The statement of the proposition will follow from this and scale invariance. Since the estimate of Proposition 3.4 depends on the embedding of the γ -quantum cone into \mathbb{C} , a key technical difficulty in the proof is to control the change in the embedding when we translate from 0 to $\eta(x)$ for $x \in \epsilon\mathbb{Z}$. The reader may want to consult the caption of Figure 8 to see how all of the various lemmas in this subsection fit together.

We start by dealing with the γ -log singularity of h at 0.

Lemma 3.8. *Suppose h is a circle average embedding of our γ -quantum cone (Section 2.1.1). Let X be the line segment from 0 to some deterministic point of $\partial\mathbb{D}$. There is a $q > 0$ depending only on γ such that the following is true. For $\epsilon \in (0, 1)$, let N^ϵ be the number of cells of the form $\eta([x - \epsilon, x])$ for $x \in \epsilon\mathbb{Z}$ needed to cover X (as in (3.11)). Then for $u \in (0, 1)$, we have*

$$\mathbb{P}[N^\epsilon > \epsilon^{-1/d_- - u}] \preceq \epsilon^{qu^2}$$

with the implicit constant depending only on γ and u .

Proof. By [HS16, Proposition 6.2] and Lemma 3.10, we can find $a > 0$ and $q_1 > 0$ such that

$$\mathbb{P}[B_{\epsilon^a}(0) \not\subset \eta([- \epsilon, \epsilon])] \preceq \epsilon^{q_1}. \quad (3.21)$$

Hence we only need to cover $X \setminus B_{\epsilon^a}(0)$. We do this using a multi-scale argument.

Let $h^G := h + \gamma \log |\cdot|$, so that by our choice of embedding $h^G|_{\mathbb{D}}$ agrees in law with the restriction to \mathbb{D} of a whole-plane GFF. For $r > 0$, let $h_r^G(0)$ be the circle average of h^G over $\partial B_r(0)$. For $k \in \mathbb{N}_0$, let

$$h^k := h^G(e^{-k} \cdot) - \gamma \log |\cdot| - h_{e^{-k}}^G(0).$$

By conformal invariance of the whole-plane GFF (modulo additive constant) we have $h^k|_{\mathbb{D}} \stackrel{d}{=} h|_{\mathbb{D}}$. Let η^k be given by $e^k \eta$, parametrized by μ_{h^k} instead of μ_h . Also let

$$X_k := X \cap (B_{e^{-k}}(0) \setminus B_{e^{-k-1}}(0)).$$

Let $v > 0$ (to be chosen later, depending on u) and let E_k be the event that the following is true.

1. $|h_{e^{-k}}^G(0)| \leq \frac{v}{\gamma} \log(\epsilon^{-1})$.
2. There exists a collection \mathcal{I}_k of at most $\epsilon^{-(1/d_- + v)(1+v)}$ intervals of length at most $\frac{1}{2}\epsilon^{1+v}$ such that $\bigcup_{I \in \mathcal{I}_k} \eta^k(I)$ covers X_0 .

The random variable $h_{e^{-k}}^G(0)$ is Gaussian with variance k [DS11, Section 3.1], so by the Gaussian tail bound and Proposition 3.4, we have

$$\mathbb{P}[E_k^c] \preceq \epsilon^{q_2 v^2}, \quad \forall k \in [0, \lceil \log \epsilon^{-a} \rceil]_{\mathbb{Z}}$$

for appropriate $q_2 > 0$ depending only on γ . Therefore,

$$\mathbb{P}\left[\bigcap_{k=0}^{\lceil \log \epsilon^{-a} \rceil} E_k\right] \geq 1 - \epsilon^{q_2 v^2 + o_\epsilon(1)} \quad (3.22)$$

Now suppose that $\bigcap_{k=0}^{\lceil \log \epsilon^{-a} \rceil} E_k$ occurs. By [DS11, Proposition 2.1], for each $k \in \mathbb{N}_0$ and each $A \subset \mathbb{D}$ we have

$$\mu_{h^k}(A) = \exp\left(k\left(2 - \frac{\gamma^2}{2}\right) - \gamma h_{e^{-k}}^G(0)\right) \mu_h(e^{-k}A).$$

In particular, if E_k occurs and $I \in \mathcal{I}_k$, then

$$\frac{1}{2}\epsilon^{1+v} \geq \text{len } I = \mu_{h^k}(\eta^k(I)) \geq e^{(2-\gamma^2/2)k} \epsilon^{-v} \mu_h(e^{-k}\eta^k(I)).$$

Hence $e^{-k}\eta^k(I) \subset \eta(J)$ for an interval $J \subset \mathbb{R}$ with length at most ϵ . If we let \mathcal{J}_k be the collection of all such intervals J , then $\bigcup_{J \in \mathcal{J}_k} \eta(J)$ covers $e^{-k}X_0 = X_k$. Therefore,

$$X \setminus B_{\epsilon^a}(0) \subset \bigcup_{k=0}^{\lceil \log \epsilon^{-a} \rceil} \bigcup_{J \in \mathcal{J}_k} \eta(J).$$

The total number of intervals in $\bigcup_{k=0}^{\lceil \log \epsilon^{-a} \rceil} \mathcal{J}_k$ is at most $\log \epsilon^{-a} \epsilon^{-(1/d_- + v)(1+v)}$. If we take $v = cu$ for an appropriate $c = c(\gamma) > 0$, then this quantity is smaller than $\frac{1}{2}\epsilon^{-1/d_- - u}$ for small enough ϵ . Recalling (3.21) and (3.22), we obtain the statement of the lemma with $q = \min\{q_1, q_2 c^2\}$. \square

Lemma 3.9. *Suppose h is a circle average embedding of our quantum cone. For $\epsilon > 0$, $u > 0$, and $r \in (0, 1)$, let $E_\epsilon(r) = E_\epsilon(r, u)$ be the event that the following is true. For each $x \in \epsilon\mathbb{Z}$ such that $\eta([x - \epsilon, x])$ intersects $\partial B_r(0)$, we have*

$$\text{dist}(0, x; \mathcal{G}^\epsilon) \leq \epsilon^{-1/d_- - u}.$$

There exists $q > 0$ depending only on γ such that for each $u > 0$ and each $r \in (0, 1/2]$, we have

$$\mathbb{P}[E_\epsilon(r)^c] \preceq \epsilon^{qu^2}$$

with the implicit constant depending only on γ and u (not on r).

Proof. By Proposition 3.4 and a scaling argument as in the proof of Lemma 3.8, it holds except on an event of probability $\epsilon^{q_0 u^2}$ for $q_0 = q_0(\gamma) > 0$ that the number of cells $\eta([y - \epsilon, y])$ for $y \in \epsilon\mathbb{Z}$ needed to cover $\partial B_r(0)$ is at most $\frac{1}{2}\epsilon^{-1/d_- - u}$. The lemma follows by combining this with Lemma 3.8. \square

We next want to use translation invariance to obtain an analogue of Lemma 3.9 with $\eta(x)$ for appropriate $x \in \epsilon\mathbb{Z}$ in place of $0 = \eta(0)$. By [DMS14, Lemma 9.3], we know that $(h(\cdot + \eta(t)), \eta(\cdot + t) - \eta(t)) \stackrel{d}{=} (h, \eta)$ for each $t > 0$, modulo embedding. Since the statement of Lemma 3.9 depends on the choice of embedding, we need some lemmas to control how much the embedding of $(h(\cdot + \eta(t)), \eta(\cdot + t) - \eta(t))$ differs from a circle average embedding. This will be accomplished by estimating the quantum mass of certain balls under each embedding.

Lemma 3.10. *Let $\alpha \in (0, Q)$ (with Q as in (2.1)) and let h be a circle average embedding of an α -quantum cone in $(\mathbb{C}, 0, \infty)$. For $p > 0$ and $\epsilon \in (0, 1)$, we have*

$$\mathbb{P}\left[\mu_h(B_\epsilon(0)) > \epsilon^{2 + \frac{\gamma^2}{2} - \alpha\gamma - p}\right] \leq \epsilon^{\frac{2}{\gamma^2}(\sqrt{4+2p}-2)^2 + o_\epsilon(1)}.$$

Proof. Let $h^G := h + \alpha \log |\cdot|$, so that by our choice of embedding $h^G|_{\mathbb{D}}$ agrees in law with the restriction to \mathbb{D} of a whole-plane GFF. For $r > 0$, let $h_r^G(0)$ be the circle average of h^G over $\partial B_r(0)$. Also let $h^{G,r} := h^G(r\cdot) - h_r^G(0)$. Then $h^{G,r}|_{\mathbb{D}} \stackrel{d}{=} h^G|_{\mathbb{D}}$.

For $k \in \mathbb{N}_0$ let A_k be the annulus $B_{e^{-k}}(0) \setminus B_{e^{-k-1}}(0)$. By [DS11, Proposition 2.1],

$$\mu_h(A_k) = \exp\left(-k\left(2 + \frac{\gamma^2}{2} - \alpha\gamma\right) + \gamma h_{e^{-k}}^G(0)\right) \int_{A_0} |z|^{-\alpha\gamma} d\mu_{h^{G,e^{-k}}}(z).$$

The random variable $h_{e^{-k}}^G(0)$ is Gaussian with variance k [DS11, Section 3.1], so for $p > 0$ we have

$$\mathbb{P}[\exp(\gamma h_{e^{-k}}^G(0)) > e^{kp}] \preceq e^{-\frac{p^2}{2\gamma^2}k}.$$

Furthermore, since

$$\int_{A_0} |z|^{-\alpha\gamma} d\mu_{h_{G,e^{-k}}}(z) \preceq \mu_{h_{G,e^{-k}}}(\mathbb{D})$$

has finite moments of all orders $< 4/\gamma^2$ [RV14, Theorem 2.11], we have

$$\mathbb{P}\left[\int_{A_0} |z|^{-\alpha\gamma} d\mu_{h_{G,e^{-k}}}(z) > e^{kp}\right] \leq e^{-\left(\frac{4p}{\gamma^2} + o_k(1)\right)k}.$$

Hence for $s \in (0, 1)$,

$$\mathbb{P}\left[\mu_h(A_k) > \exp\left(-k\left(2 + \frac{\gamma^2}{2} - \alpha\gamma - p\right)\right)\right] \leq \exp\left(-k\left(\frac{p^2 s^2}{2\gamma^2} \wedge \frac{4p(1-s)}{\gamma^2} + o_k(1)\right)\right).$$

Optimizing over s gives

$$\mathbb{P}\left[\mu_h(A_k) > \exp\left(-k\left(2 + \frac{\gamma^2}{2} - \alpha\gamma - p\right)\right)\right] \leq \exp\left(-k\left(\frac{2}{\gamma^2}(\sqrt{4+2p}-2)^2 + o_k(1)\right)\right).$$

Given $\epsilon \in (0, 1)$, let $k \in \mathbb{N}_0$ be chosen so that $2^{-k-1} \leq \epsilon \leq 2^{-k}$. For an appropriate k -independent constant $c > 0$, we have

$$\begin{aligned} \mathbb{P}\left[\mu_h(B_\epsilon(0)) > \epsilon^{2+\frac{\gamma^2}{2}-\alpha\gamma-p}\right] &\leq \sum_{j=k}^{\infty} \mathbb{P}\left[\mu_h(A_j) > c \exp\left(-j\left(2 + \frac{\gamma^2}{2} - \alpha\gamma - p\right)\right)\right] \\ &\leq \sum_{j=k}^{\infty} \exp\left(-j\left(\frac{2}{\gamma^2}(\sqrt{4+2p}-2)^2 + o_j(1)\right)\right) \\ &\leq \epsilon^{\frac{2}{\gamma^2}(\sqrt{4+2p}-2)^2 + o_\epsilon(1)}. \end{aligned} \quad \square$$

Lemma 3.11. *Let $\alpha \in (0, Q)$ and let h be a circle average embedding of an α -quantum cone in $(\mathbb{C}, 0, \infty)$. Then for each fixed $r \in (0, 1]$ and each $\epsilon \in (0, 1)$,*

$$\mathbb{P}[\mu_h(B_r(0)) < \epsilon] = o_\epsilon^\infty(\epsilon) \quad (3.23)$$

at a rate depending on r .

Proof. By our choice of embedding the law of $h|_{\mathbb{D}}$ is the same as the law of the restriction of $h^G - \alpha \log |\cdot|$ to \mathbb{D} , where h^G is a whole-plane GFF normalized so its circle average over $\partial\mathbb{D}$ is 0. Hence $\mu_h(B_r(0))$ stochastically dominates $\mu_{h^G}(B_r(0))$. It is easy to see from [DS11, Lemma 4.5] (see, e.g., the proof of [GHM15, Lemma 3.8]) that $\mathbb{P}[\mu_{h^G}(B_r(0)) < \epsilon^{1/2} e^{\gamma h_r^G(0)}] = o_\epsilon^\infty(\epsilon)$, where $h_r^F(0)$ is the circle average of h^G over $\partial B_r(0)$. Since $h_r^F(0)$ is Gaussian with variance $\log r^{-1}$, we also have $\mathbb{P}[e^{\gamma h_r^G(0)} < \epsilon^{1/2}] = o_\epsilon^\infty(\epsilon)$. \square

The following lemma will allow us to compare the embedding $h(\cdot + \eta(t))$ with a circle average embedding of the γ -quantum cone $(\mathbb{C}, h, \eta(t), 0)$.

Lemma 3.12. *Let (h, η) be as in Section 1.3 and assume that h is a circle average embedding. For $t \in \mathbb{R}$, let $h^t = h(\cdot + \eta(t))$ so that by [DMS14, Lemma 9.3], we have $(\mathbb{C}, h^t, 0, \infty) \stackrel{d}{=} (\mathbb{C}, h, 0, \infty)$, modulo embedding. Let $\rho_t \in \mathbb{C}$ be a random variable chosen so that $h^t(\rho_t^{-1} \cdot) + Q \log |\rho_t^{-1}|$ is a circle average embedding of the quantum cone $(\mathbb{C}, h^t, 0, \infty)$ (so that $|\rho_t|$ is a deterministic functional of h^t ; see Section 2.1.1). There exists $a, q > 0$ depending only on γ such that for each $\epsilon \in (0, 1)$ and each $u \in (0, 1)$, we have*

$$\mathbb{P}[\eta([0, \epsilon^u]) \subset B_{\epsilon^{au}}(0) \text{ and } |\rho_x| > 2|\eta(x)|, \forall x \in (0, \epsilon^u]_{\mathbb{Z}}] \geq 1 - O_\epsilon(\epsilon^{qu}).$$

Proof. Let $a > 0$ to be chosen later and let τ_ϵ be the exit time of η from $B_{\epsilon^{au}}(0)$. By [GHM15, Lemma 3.4], except on an event of probability $o_\epsilon^\infty(\epsilon)$ it holds that $\eta([0, \tau_\epsilon])$ contains a Euclidean ball of radius at least ϵ^{2au} . Note that $\eta([0, \tau_\epsilon])$ is independent from h . By [GHM15, Lemma 3.8], if a is chosen sufficiently small (depending only on γ) then we can find $q_1 > 0$ such that the probability that this Euclidean ball has quantum mass smaller than ϵ^u is $\preceq \epsilon^{q_1 u^2}$. Hence

$$\mathbb{P}[\eta([0, \epsilon^u]) \subset B_{\epsilon^{au}}(0)] \geq 1 - O_\epsilon(\epsilon^{q_1 u^2}). \quad (3.24)$$

By Lemma 3.10, if we fix $b \in (0, 2 - \gamma^2)$ then

$$\mathbb{P}[\mu_h(B_{2\epsilon^{au}}(0)) \leq \epsilon^{abu}] \geq 1 - O_\epsilon(\epsilon^{q_2 u}) \quad (3.25)$$

for appropriate $q_2 = q_2(\gamma, b) > 0$.

By [DS11, Proposition 2.1], with ρ_t as in the statement of the lemma, it holds that $\mu_{h^t}(B_{|\rho_t|/2}(0)) = \mu_h(B_{|\rho_t|/2}(\eta(t)))$ has the same law as $\mu_h(\mathbb{D})$. By Lemma 3.11,

$$\mathbb{P}[\mu_h(B_{|\rho_t|/2}(\eta(t))) < \epsilon^{abu}] = o_\epsilon^\infty(\epsilon). \quad (3.26)$$

Hence if $\mu_h(B_{2\epsilon^{au}}(0)) \leq \epsilon^{abu}$ it holds except on an event of probability $o_\epsilon^\infty(\epsilon)$ that $B_{|\rho_t|/2}(\eta(t)) \not\subset B_{2\epsilon^{au}}(0)$. So, if $t \in [0, \epsilon^u]$, then on the event that $\eta([0, \epsilon^u]) \subset B_{\epsilon^{au}}(0)$ and $\mu_h(B_{2\epsilon^{au}}(0)) \leq \epsilon^{abu}$ it holds except on an event of probability $o_\epsilon^\infty(\epsilon)$ that $|\rho_t| > 2|\eta(t)|$. We obtain the statement of the lemma with $q = q_1 \wedge q_2$ by combining (3.24) and (3.25) and applying the union bound over all $x \in (0, \epsilon^u]_{\epsilon\mathbb{Z}}$. \square

Proof of Proposition 3.7. See Figure 8 for an illustration of the proof. For $x \in \epsilon\mathbb{Z}$, let $h^x = h(\cdot + \eta(x))$ and $\rho_x \in \mathbb{C}$ be as in Lemma 3.12 and let $\eta^x := \eta(\cdot + x) - \eta(x)$. By [DMS14, Lemma 9.3], we have $(h^x, \eta^x) \stackrel{d}{=} (h, \eta)$, modulo embedding. Let $a > 0$ be as in Lemma 3.12 and let E_ϵ^* be the event $E_\epsilon(\epsilon^{au})$ from Lemma 3.9 (with $r = \epsilon^{au}$). Also let E_ϵ^x be the event $E_\epsilon(1/2)$ from Lemma 3.9 with (h^x, η^x) in place of (h, η) .

Suppose now that $x \in \epsilon\mathbb{Z}$ and the event

$$E_\epsilon^x \cap E_\epsilon^* \cap \{\eta(x) \in B_{\epsilon^{au}}(0)\} \cap \{|\rho_x| > 2|\eta(x)|\} \quad (3.27)$$

occurs. By definition of E_ϵ^x , the distance from x to each $y \in \epsilon\mathbb{Z}$ such that $\eta([y - \epsilon, y])$ intersects $\partial B_{|\rho_x|/2}(\eta(x))$ is at most $\epsilon^{-1/d_- - u}$. Furthermore, since each of the balls $B_{\epsilon^{au}}(0)$ and $B_{|\rho_x|/2}(\eta(x))$ contains the center of the other, there is some such vertex y such that any path in \mathcal{G}^ϵ from x to y must pass through some $z \in \epsilon\mathbb{Z}$ such that $\eta([z - \epsilon, z])$ intersects $\partial B_{\epsilon^{au}}(0)$. By definition of E_ϵ^* , we thus have $\text{dist}(0, x; \mathcal{G}^\epsilon) \leq 2\epsilon^{-1/d_- - u}$.

It follows from Lemmas 3.9 and 3.12 that there is a $q > 0$ depending only on γ such that for each $x \in (0, \epsilon^u]_{\epsilon\mathbb{Z}}$, it holds with probability at least $1 - O_\epsilon(\epsilon^{qu^2})$ that the event (3.27) occurs, with the $O_\epsilon(\epsilon^{qu^2})$ uniform over all $x \in (0, \epsilon^u]_{\epsilon\mathbb{Z}}$. Hence

$$\mathbb{P}[\text{dist}(0, x; \mathcal{G}^\epsilon) \leq 2\epsilon^{-1/d_- - u}] \geq 1 - O_\epsilon(\epsilon^{qu^2}).$$

By the Chebyshev inequality, with probability at least $1 - O_\epsilon(\epsilon^{qu^2})$, there are at least $(1 - o_\epsilon(1))\epsilon^{-(1-u)}$ elements of $\epsilon\mathbb{Z}$ whose distance to 0 in \mathcal{G}^ϵ is at most $2\epsilon^{-1/d_- - u}$.

Given $n \in \mathbb{N}$, choose $\epsilon > 0$ such that $n = \lfloor 2\epsilon^{-1/d_- - u} \rfloor$, so that $\epsilon \asymp n^{\frac{d_-}{1-d_- - u}}$. Then the preceding paragraph implies that there is a $p_0 = p_0(\gamma) > 0$ such that if u is chosen sufficiently small, then except on an event of probability at most $O_n(n^{-p_0 u^2})$ there are at least $(1 - o_n(1))n^{\frac{d_-(1-u)}{1-d_- - u}}$ elements x of $\epsilon\mathbb{Z}$ with $\text{dist}(0, x; \mathcal{G}^\epsilon) \leq n$. By scale invariance, the law of $\#\mathcal{B}_n(0; \mathcal{G}^\epsilon)$ does not depend on ϵ . The statement of the lemma for small enough u follows by replacing u with cu where $c = c(\gamma)$ is chosen so that $(1 - o_n(1))n^{\frac{d_-(1-cu)}{1-d_- - cu}} \leq n^{d_- - u}$ for small enough u . The statement for general $u \in (0, 1)$ follows by shrinking p . \square

Proof of Theorem 1.11. The upper bound follows from (3.1) of Proposition 3.1 and scale invariance. The lower bound follows from Proposition 3.7. \square

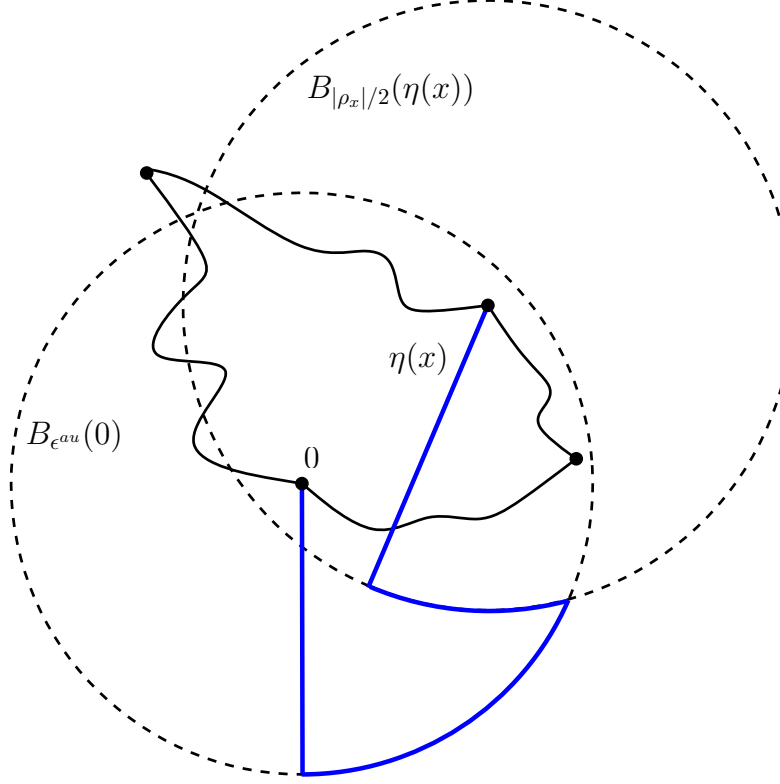


Figure 8: An illustration of the proof of Proposition 3.7. Lemma 3.9 implies that with high probability, the distance in \mathcal{G}^ϵ from 0 to any cell which intersects the circle $\partial B_{\epsilon^{au}}(0)$, with a as in Lemma 3.12 is at most $\epsilon^{-1/d-u}$. Lemma 3.12 implies that if $x \in (0, \epsilon^u]_{\mathbb{Z}}$, then with high probability $\eta(x) \in B_{\epsilon^{au}}(0)$ and $0 \in B_{|\rho_x|/2}(\eta(x))$. The ball $B_{|\rho_x|/2}(\eta(x))$ is mapped to $B_{1/2}(0)$ when we re-scale to get a circle average embedding of the quantum cone $(\mathbb{H}, h, \eta(x), \infty)$. Hence Lemma 3.9 implies that with high probability the distance from x to any point in $\partial B_{|\rho_x|/2}(\eta(x))$ is at most $\epsilon^{-1/d-u}$. We obtain a bound on $\text{dist}(0, x; \mathcal{G}^\epsilon)$ by concatenating a path from x to an appropriate point of $\partial B_{|\rho_x|/2}(\eta(x))$ with the reverse of a path from 0 to an appropriate point of $\partial B_{\epsilon^{au}}(0)$ (the concatenated path is shown in blue).

Remark 3.13 (Upper bound for χ). In this remark we describe what is needed to extract the upper bound for the exponent χ of Theorem 1.12 described in Remark 1.14 from the results of this subsection and the other estimates in this paper.

We first note that if the lower bound of Theorem 1.15 held for distances in \mathcal{G}^ϵ rather than in $\mathcal{G}^\epsilon|_{(0,1]}$ (which we expect to be the case for $\gamma \in (0, \gamma_*]$, as defined in Conjecture 1.13) then the upper bound for χ would follow from Proposition 3.4, applied with X equal to a straight line, plus a similar (but easier) argument to the one used to prove Proposition 3.7.

In order to extract an upper bound for χ using only the results of this paper, we would need to apply Proposition 3.4 to a set X which is contained in $\eta([0, \tau])$ for an appropriate choice of time τ . We can reduce to the case when τ is a stopping time depending only on η , viewed modulo parametrization, by similar arguments to the ones used earlier in this subsection. Due to our strong upper bound for distances ((1.12) of Theorem 1.15) we only need to consider paths between points near $\eta(0)$ and $\eta(\tau)$, not between $\eta(0)$ and $\eta(\tau)$ themselves. Hence we would need an upper bound for the minimal Euclidean length of a curve between appropriate points of $\eta([0, \tau])$ if we restrict to curves which are contained in $\eta([0, \tau])$.

In the case when $\kappa \geq 8$ (so that the interior of $\eta([0, \tau])$ is connected) we expect that one can prove such a bound as follows. First conformally map $\eta([0, \infty))$ to \mathbb{H} and apply distortion estimates to reduce to the problem of bounding the length of a Euclidean curve contained in a segment of a chordal SLE_κ curve. By considering a curve which stays close (but not too close) to \mathbb{R} and applying SLE duality (in

particular [Zha08, Theorem 5.2]), we reduce to the problem of estimating how close a certain $\text{SLE}_\kappa(\underline{\rho})$ curve in \mathbb{H} gets to a given point in \mathbb{R} . Such an estimate for $\text{SLE}_\kappa(\underline{\rho})$ curves with a single force point is proven in [Law15, Proposition 5.4], and this estimate can be extended to the case of multiple force points (which arises in SLE duality) using conditioning arguments as in [MS16d, Section 7]. The case when $\kappa \in (4, 8)$ can be treated similarly, except we need to apply an argument like the one above to construct a path in each connected component of the interior of $\eta([0, \tau])$.

4 Expected diameter of a cell conditioned on its boundary lengths

Fix $\gamma \in (0, 2)$ and assume we are in the setting of Section 1.3. In this section we will prove an estimate which shows that the conditional expected diameter of $\mathcal{G}^{2^{-n}}|_{(0,1]}$ given a realization of the boundary length vector $\Delta_{[0,1]}^Z$ (Definition 2.2) with $|\Delta_{[0,1]}^Z|$ not too large is not too much larger than its unconditional expected diameter.

Proposition 4.1. *Let $n \in \mathbb{N}$ and let F_n be as in Lemma 2.6. For each $a = (\underline{a}_L, \underline{a}_R, \bar{a}_L, \bar{a}_R) \in (0, \infty)^4$ with $|\Delta_{[0,1]}^Z| \leq n$, we have*

$$\mathbb{E} \left[\text{diam} \left(\mathcal{G}^{2^{-n}}|_{(0,1]} \right) \mathbb{1}_{F_n} \mid \Delta_{(0,1]}^Z = a \right] \preceq n^5 \mathbb{E} \left[\text{diam} \mathcal{G}^{2^{-n}}|_{(0,1]} \right] \quad (4.1)$$

with the implicit constant depending only on γ .

The reason why Proposition 4.1 is useful is as follows. By Lemma 2.3, the graph $\mathcal{G}^{2^{-n}}|_{(0,1]}$ is determined by $\{\Delta_{[x-2^{-n}, x]}^Z : x \in (0, 1]_{2^{-n}\mathbb{Z}}\}$. Hence for $n, m \in \mathbb{N}$, Proposition 4.1 allows us to estimate the conditional expected diameter given $\{\Delta_{[x-2^{-n}, x]}^Z : x \in (0, 1]_{2^{-n}\mathbb{Z}}\}$ of $\mathcal{G}^{2^{-n-m}}|_{(x-2^{-n}, x]}$ for $x \in (0, 1]_{2^{-n}\mathbb{Z}}$. Such an estimate will play a key role in the next section (see in particular Lemma 5.9).

To prove Proposition 4.1, we will start in Section 4.1 by proving an analogous estimate when we condition on only $Z_1 = (\bar{\Delta}_{(0,1]}^L - \underline{\Delta}_{(0,1]}^L, \bar{\Delta}_{(0,1]}^R - \underline{\Delta}_{(0,1]}^R)$ instead of on the whole boundary length vector $\Delta_{(0,1]}^Z$. In Section 4.2, we will improve this to an estimate where we condition on Z_1 and the event that the infimum of L (resp. R) on $[0, 1]$ is at least $-b_L$ (resp. $-b_R$) for some $b_L, b_R \geq 0$, but not on the precise values of these infima. In Section 4.3, we will conclude the proof of Proposition 4.1.

4.1 Conditioning on just the endpoints

In this subsection we will prove a weaker version of Proposition 4.1 in which we condition only on $Z_1 = (\bar{\Delta}_{(0,1]}^L - \underline{\Delta}_{(0,1]}^L, \bar{\Delta}_{(0,1]}^R - \underline{\Delta}_{(0,1]}^R)$ instead of on $\Delta_{(0,1]}^Z$. In this case, the proof of the proposition amounts to an elementary Radon-Nikodym calculation for a Brownian bridge.

Lemma 4.2. *Let $\epsilon > 0$ and $w \in \mathbb{R}^2$. Also let $n \in \mathbb{N}$ such that $2^{-n} \leq \epsilon$. Then*

$$\mathbb{E} \left[\text{diam} \left(\mathcal{G}^\epsilon|_{(0,1]} \right) \mid Z_1 = w \right] \preceq (1 \vee |w|)^2 (\log \epsilon^{-1})^2 \mathbb{E} \left[\text{diam} \left(\mathcal{G}^{2^{-n}}|_{(0,1]} \right) \right] \quad (4.2)$$

with the implicit constant depending only on γ .

Proof. Let Σ be the covariance matrix of Z . Then for each $t_1, t_2 \in [0, 1]$ with $t_2 - t_1 = \delta > 0$, the unconditional density of $Z_{t_2} - Z_{t_1}$ is given by

$$z \mapsto \frac{1}{2\pi\sqrt{\delta \det \Sigma}} \exp \left(-\frac{\langle z, \Sigma^{-1}z \rangle}{2\delta} \right) \quad (4.3)$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on \mathbb{R}^2 . Furthermore, by a straightforward Gaussian calculation, the regular conditional law of $Z_{t_2} - Z_{t_1}$ given $\{Z_1 = w\}$ is bivariate Gaussian with mean δw and covariance matrix $\delta(1 - \delta)\Sigma$, i.e. the density of this regular conditional law with respect to Lebesgue measure is given by

$$z \mapsto \frac{1}{2\pi\sqrt{\delta(1 - \delta) \det \Sigma}} \exp \left(-\frac{\langle z - \delta w, \Sigma^{-1}(z - \delta w) \rangle}{2\delta(1 - \delta)} \right). \quad (4.4)$$

The ratio of the above two densities gives the Radon-Nikodym derivative of the conditional law of $Z_{t_2} - Z_{t_1}$ with respect to its unconditional law. By the Markov property, this is the same as the Radon-Nikodym derivative of the conditional law of $\{Z_t - Z_{t_1} : t \in [t_1, t_2]\}$ given $\{Z_1 = w\}$ with respect to its marginal law. If $\delta \leq 1/2$, this Radon-Nikodym derivative is at most

$$2 \exp \left(\frac{2\langle z, \Sigma^{-1}w \rangle - \langle z, \Sigma^{-1}z \rangle - \delta \langle w, \Sigma^{-1}w \rangle}{2(1-\delta)} \right). \quad (4.5)$$

Let $K > 1$ be a constant (depending only on γ) such that

$$K^{-1}|z|^2 \leq \langle z, \Sigma^{-1}z \rangle \leq K|z|^2, \quad \forall z \in \mathbb{R}^2.$$

By the Gaussian tail bound and the form of the density (4.4), we find that for $C > 0$,

$$\mathbb{P} \left[|Z_{t_2} - Z_{t_1}| > (2KC\delta)^{1/2} + \delta|w| \mid Z_1 = w \right] \preceq e^{-C} \quad (4.6)$$

with the implicit constant depending only on γ . Furthermore, whenever $|z| \leq (2KC\delta)^{1/2} + \delta|w|$, the quantity (4.5) is at most

$$2 \exp \left(\frac{K((2KC\delta)^{1/2} + \delta|w|)|w|}{2(1-\delta)} \right). \quad (4.7)$$

Now suppose we are given $\epsilon > 0$ and $w \in \mathbb{R}^2$. In the above estimates, take $C = \log \epsilon^{-1}$ and let $t_1, t_2 \in [0, 1]$ be chosen so that

$$\delta = t_2 - t_1 = (1 \vee |w|)^{-2} (\log \epsilon^{-1})^{-1}. \quad (4.8)$$

Let E be the event that $|Z_{t_2} - Z_{t_1}| \leq (2K\delta \log \epsilon^{-1})^{1/2} + \delta|w|$. By (4.6) with this choice of C and δ we have $\mathbb{P}[E^c \mid Z_1 = w] \preceq \epsilon$. By (4.7), on E the Radon-Nikodym derivative of the conditional law of $\{Z_t - Z_{t_1} : t \in [t_1, t_2]\}$ given $\{Z_1 = w\}$ with respect to its marginal law is at most a constant depending only on K . The graph $\mathcal{G}^\epsilon|_{[t_1, t_2]}$ is determined by $\{Z_t - Z_{t_1} : t \in [t_1, t_2]\}$ and the diameter of this graph is at most ϵ^{-1} . Hence for any $\delta \in (0, 1/2]$,

$$\begin{aligned} \mathbb{E}[\text{diam}(\mathcal{G}^\epsilon|_{[t_1, t_2]}) \mid Z_1 = w] &\leq \mathbb{E}[\text{diam}(\mathcal{G}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_E \mid Z_1 = w] + \epsilon^{-1} \mathbb{P}[E^c \mid Z_1 = w] \\ &\preceq \mathbb{E}[\text{diam}(\mathcal{G}^\epsilon|_{[0, t_2 - t_1]})] \\ &\leq \log \epsilon^{-1} \mathbb{E}[\text{diam}(\mathcal{G}^{2^{-n}}|_{(0, 1]})], \end{aligned} \quad (4.9)$$

where in the last inequality we have used Lemmas 2.4 and 2.5. If we write $(0, 1]$ as the union of $\lceil \delta^{-1} \rceil \asymp (1 \vee |w|)^2 \log \epsilon^{-1}$ intervals of length δ , then $\text{diam}(\mathcal{G}^\epsilon|_{(0, 1]})$ is at most the sum of the diameters of the restrictions of \mathcal{G}^ϵ to these intervals. We obtain (4.2) by summing (4.9) over all of the intervals in this union. \square

4.2 Estimates for conditioned Brownian bridge

In this subsection we will prove an estimate which will serve as an intermediate step between Lemma 4.2 and Proposition 4.1. Namely, we will bound the conditional expected diameter of the structure graph over $(0, 1]$ when we condition on Z_1 and on lower bounds for the infima of L and R on $[0, 1]$ (but not the precise values of these infima).

Let $b = (b_L, b_R) \in \mathbb{R}^2$ with $b_L, b_R \geq 0$, and $w = (w_L, w_R) \in \mathbb{R}^2$ with $w_L \geq -b_L$ and $w_R \geq -b_R$. Let $\tilde{Z} = (\tilde{L}, \tilde{R})$ have the law of a correlated Brownian bridge from 0 to w in time 1 with the same variances and covariances as Z (recall (2.2)), conditioned on the event that

$$\inf_{t \in [0, 1]} \tilde{L}_t \geq -b_L \quad \text{and} \quad \inf_{t \in [0, 1]} \tilde{R}_t \geq -b_R. \quad (4.10)$$

If at least one of $b_L, b_R, w_L + b_L$, or $w_R + b_R$ is 0, then the event (4.12) has probability zero. However, one can still make sense of the law of Z , and in each case that law of Z_t for each $t \in (0, 1)$ is absolutely continuous with respect to Lebesgue measure. In particular, we have the following.

- In the case when b_L and $w_L + b_L$ are non-zero but b_R and $w_R + b_R$ are possibly zero, the law of \tilde{Z} is that of a correlated two-dimensional Brownian bridge conditioned to stay in the upper half plane, conditioned on the positive probability event that its first coordinate stays above $-b_L$. This law can be obtained by applying a linear transformation to a pair consisting of a one-dimensional Brownian bridge and an independent one-dimensional Brownian bridge conditioned to stay positive, conditioned on a certain positive probability event. A similar statement holds with “ L ” and “ R ” interchanged.
- In the case when b_L and $w_R + b_R$ are non-zero but b_R and $w_L + b_L$ are zero, the law of $\tilde{Z}|_{[0,1/2]}$ is that of a correlated Brownian motion conditioned to stay in the upper half plane and conditioned on the positive probability event that its first coordinate stays above $-b_L$, weighted by a smooth function. The conditional law of the time reversal of $\tilde{Z}|_{[1/2,1]}$ given $\tilde{Z}|_{[0,1/2]}$ is that of a correlated Brownian bridge conditioned to stay in the upper half plane. A similar statement holds with “ L ” and “ R ” interchanged.
- In the case when $b_L = b_R = 0$ but $w_L + b_L$ and $w_R + b_R$ are non-zero, the law of \tilde{Z} is that of a correlated two-dimensional Brownian bridge conditioned to stay in the first quadrant. This law is rigorously defined, e.g., in [GS15a, Section 1.3.1] or [DW15a], building on [Shi85] (which constructs a correlated Brownian motion conditioned to stay in the first quadrant). The same applies to the time reversal of \tilde{Z} in the case when $w_L + b_L = w_R + b_R = 0$ but b_L and b_R are non-zero.
- In the case when at least three of b_L , b_R , $w_L + b_L$, and $w_R + b_R$ are zero, either \tilde{Z} or its time reversal has the law of a correlated Brownian $\pi/2$ -cone excursion conditioned to spend one unit of time in the cone at a particular point. See [MS15c, Section 3] or [DW15b] for more detail.

For $\epsilon > 0$, let $\tilde{\mathcal{G}}^\epsilon$ be defined in the same manner as the structure graph $\mathcal{G}^\epsilon|_{(0,1]}$ with \tilde{Z} in place of $Z|_{[0,1]}$. For $C \geq |w|$, define the regularity event

$$\tilde{F}_C := \left\{ \sup_{s,t \in [0,1]} |\tilde{Z}_s - \tilde{Z}_t| \leq C \right\}. \quad (4.11)$$

The main result of this subsection is the following lemma.

Lemma 4.3. *For each choice of b, w, t_1, t_2 as above, each $C > |w|$, each $\epsilon > 0$, and each $n \in \mathbb{N}$ with $2^{-n} \leq \epsilon$, we have*

$$\mathbb{E} \left[\text{diam}(\tilde{\mathcal{G}}^\epsilon) \mathbb{1}_{\tilde{F}_C} \right] \preceq (1 \vee C)^2 (\log \epsilon^{-1})^2 n \mathbb{E} \left[\text{diam}(\mathcal{G}^{2^{-n}}|_{(0,1]}) \right]$$

with the implicit constant depending only on γ .

For the proof of Lemma 4.3, we need the following estimate for a slightly different conditioned Brownian motion. The lemma will eventually allow us to compare $Z|_{[t_1, t_2]}$ for $0 < t_1 < t_2 < 1$ to a correlated Brownian bridge with no additional conditioning, which will in turn allow us to apply Lemma 4.2.

Lemma 4.4. *Let $b = (b_L, b_R) \in \mathbb{R}^2$ with $b_L, b_R \geq 0$ and $w = (w_L, w_R) \in \mathbb{R}^2$ with $w_L \geq -b_L$ and $w_R \geq -b_R$. Also let $t_1, t_2 \in (0, 1)$ with $0 \leq t_2 - t_1 \leq t_1 \wedge (1 - t_2)$. Let $\hat{Z} = (\hat{L}, \hat{R})$ have the law of a Brownian bridge from 0 to w in time 1, with covariance matrix Σ , conditioned on the event that*

$$\inf_{t \in [0, t_1] \cup [t_2, 1]} \hat{L}_t \geq -b_L \quad \text{and} \quad \inf_{t \in [0, t_1] \cup [t_2, 1]} \hat{R}_t \geq -b_R. \quad (4.12)$$

Let

$$\hat{E} := \left\{ \inf_{t \in [t_1, t_2]} \hat{L}_t \geq -b_L, \inf_{t \in [t_1, t_2]} \hat{R}_t \geq -b_R \right\} = \left\{ \inf_{t \in [0, 1]} \hat{L}_t \geq -b_L, \inf_{t \in [0, 1]} \hat{R}_t \geq -b_R \right\}. \quad (4.13)$$

There is a constant $c_1 > 0$ depending only on γ such that for any choice of b, w, t_1, t_2 as above, $\mathbb{P}[\hat{E}] \geq c_1$.

Proof. Let τ_1 be the smallest $t \in [0, t_1]$ for which $Z_t \in [-b_L + t_1^{1/2}, \infty) \times [-b_R + t_1^{1/2}, \infty)$ or $\tau_1 = t_1$ if no such t exists. Also let τ_2 be the largest $t \in [t_2, 1]$ for which $Z_t \in [-b_L + (1 - t_2)^{1/2}, \infty) \times [-b_R + (1 - t_2)^{1/2}, \infty)$. The regular conditional law of $\hat{Z}|_{[0, t_1]}$ given $\hat{Z}|_{[t_1, 1]}$ is that of a Brownian bridge from 0 to \hat{Z}_{t_1} in time t_1

conditioned to stay in $[-b_L, \infty) \times [-b_R, \infty)$. Such a Brownian bridge has uniformly positive probability to enter $[-b_L + t_1^{1/2}, \infty) \times [-b_R + t_1^{1/2}, \infty)$ before time t_1 , so we can find $p_1 > 0$ depending only on γ such that $\mathbb{P}[\tau_1 < t_1 \mid \widehat{Z}|_{[t_1, 1]}] \geq p_1$. Similarly, we can find $p_2 > 0$ depending only on γ such that $\mathbb{P}[\tau_2 > t_2 \mid \widehat{Z}|_{[0, t_2]}] \geq p_2$. Then $\mathbb{P}[\tau_1 < t_1, \tau_2 > t_2] \geq p_1 p_2$. The regular conditional law of $\widehat{Z}|_{[\tau_1, \tau_2]}$ given $\widehat{Z}|_{[0, \tau_1]}$ and $\widehat{Z}|_{[\tau_2, 1]}$ is that of a correlated Brownian bridge from \widehat{Z}_{τ_1} to \widehat{Z}_{τ_2} conditioned on the event that it stays in $[-b_L, \infty) \times [-b_R, \infty)$ on the time set $[\tau_1, t_1] \cup [t_2, \tau_2]$. On the event $\{\tau_1 < t_1\} \cap \{\tau_2 > t_2\}$, the first (resp. second) endpoint of this Brownian bridge lies at distance at least $t_1^{1/2}$ (resp. $(1 - t_2)^{1/2}$) from the boundary of $[-b_L, \infty) \times [-b_R, \infty)$. Since $t_2 - t_1 \leq t_1 \wedge (1 - t_2)$, it follows that

$$\mathbb{P}[\widehat{Z}|_{[t_1, t_2]} \subset [-b_L, \infty) \times [-b_R, \infty) \mid \tau_1 < t_1, \tau_2 > t_2] \geq p_3$$

for some $p_3 > 0$ depending only on γ . The statement of the lemma follows. \square

Proof of Lemma 4.3. Let $t_1, t_2 \in (0, 1)$ with $t_2 - t_1 \leq t_1 \wedge (1 - t_2)$. Let \widehat{Z} and \widehat{E} be as in Lemma 4.4 with b, w as in the lemma and our given choice of t_1, t_2 . Also let $\widehat{\mathcal{G}}^\epsilon$ be defined in the same manner as the structure graph $\mathcal{G}^\epsilon|_{[0, 1]}$ with \widehat{Z} in place of $Z|_{[0, 1]}$ and for $C > |w|^2$ let \widehat{F}_C be defined as in (4.11) with \widehat{Z} in place of \widetilde{Z} . The law of \widehat{Z} is the same as the conditional law of \widehat{Z} given \widehat{E} , so

$$\mathbb{E}[\text{diam}(\widehat{\mathcal{G}}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_{\widehat{F}_C}] = \mathbb{E}[\text{diam}(\widehat{\mathcal{G}}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_{\widehat{F}_C} \mid \widehat{E}]. \quad (4.14)$$

Let $\widehat{f} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow [0, \infty)$ be the joint density of $(\widehat{Z}_{t_1}, \widehat{Z}_{t_2})$ with respect to Lebesgue measure. Also let \widetilde{f} be the joint conditional density of $(\widehat{Z}_{t_1}, \widehat{Z}_{t_2})$ with respect to Lebesgue measure when we condition on \widehat{E} , equivalently the joint density of $(\widetilde{Z}_{t_1}, \widetilde{Z}_{t_2})$.¹ By Bayes' rule, we have

$$\widetilde{f}(z_1, z_2) = \frac{\mathbb{P}[\widehat{E} \mid (\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2)] \widehat{f}(z_1, z_2)}{\mathbb{P}[\widehat{E}]} \asymp \mathbb{P}[\widehat{E} \mid (\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2)] \widehat{f}(z_1, z_2) \quad (4.15)$$

where in the second relation we used Lemma 4.4.

Since

$$\mathbb{E}[\text{diam}(\widehat{\mathcal{G}}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_{\widehat{F}_C} \mid (\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2), \widehat{E}] \leq \frac{\mathbb{E}[\text{diam}(\widehat{\mathcal{G}}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_{\widehat{F}_C} \mid (\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2)]}{\mathbb{P}[\widehat{E} \mid (\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2)]}$$

we infer from (4.14) and (4.15) that

$$\begin{aligned} \mathbb{E}[\text{diam}(\widehat{\mathcal{G}}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_{\widehat{F}_C}] &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathbb{E}[\text{diam}(\widehat{\mathcal{G}}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_{\widehat{F}_C} \mid (\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2), \widehat{E}] \widetilde{f}(z_1, z_2) dz_1 dz_2 \\ &\leq \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathbb{E}[\text{diam}(\widehat{\mathcal{G}}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_{\widehat{F}_C} \mid (\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2)] \widehat{f}(z_1, z_2) dz_1 dz_2 \\ &= \mathbb{E}[\mathbb{E}[\text{diam}(\widehat{\mathcal{G}}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_{\widehat{F}_C} \mid (\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2)]] \end{aligned} \quad (4.16)$$

By the Markov property, the conditional law of $\widehat{Z}|_{[t_1, t_2]}$ given $\{(\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2)\}$ is that of a Brownian bridge from z_1 to z_2 . By Lemma 4.2 and scale invariance, if $|z_2 - z_1| \leq C$ then

$$\mathbb{E}[\text{diam}(\widehat{\mathcal{G}}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_{\widehat{F}_C} \mid (\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2)] \preceq (1 \vee C)^2 (\log \epsilon^{-1})^2 \mathbb{E}[\text{diam}(\mathcal{G}^{2^{-n}}|_{(0, 1]})]$$

with the implicit constant depending only on γ . On the other hand, if $(\widehat{Z}_{t_1}, \widehat{Z}_{t_2}) = (z_1, z_2)$ and $|z_2 - z_1| > C$, then \widehat{F}_C does not occur, so the above conditional probability is 0. By plugging this into (4.16), we obtain

$$\mathbb{E}[\text{diam}(\widehat{\mathcal{G}}^\epsilon|_{[t_1, t_2]}) \mathbb{1}_{\widehat{F}_C}] \preceq (1 \vee C)^2 (\log \epsilon^{-1})^2 \mathbb{E}[\text{diam}(\mathcal{G}^{2^{-n}}|_{(0, 1]})].$$

¹These densities exist in light of the discussion just after (4.10); explicit formulas are given in the papers cited there.

On the other hand we have

$$\text{diam}(\tilde{\mathcal{G}}^\epsilon) \leq \sum_{k=2}^n \text{diam}(\tilde{\mathcal{G}}^\epsilon|_{[2^{-k}, 2^{-k+1}]}) + \sum_{k=2}^n \text{diam}(\tilde{\mathcal{G}}^\epsilon|_{[1-2^{-k+1}, 1-2^{-k}]}) + 1.$$

Taking expectations yields the statement of the lemma. \square

4.3 Proof of Proposition 4.1

Let t_L (resp. t_R) be the time at which L (resp. R) attains its infimum on the interval $[0, 1]$. Also let $\mathbf{t}_L, \mathbf{t}_R \in [0, 1]$ with $\mathbf{t}_L \leq \mathbf{t}_R$ and let $r = (r_L, r_R) \in \mathbb{R}^2$ with $r_L \geq -\underline{a}_L$ and $r_R \geq -\underline{a}_R$. Let $E = E(a, \mathbf{t}_L, \mathbf{t}_R, r)$ be the event that

$$t_L = \mathbf{t}_L, \quad t_R = \mathbf{t}_R, \quad Z_{t_L} = (-\underline{a}_L, r_R), \quad Z_{t_R} = (r_L, -\underline{a}_R), \quad \text{and} \quad Z_1 = (\bar{a}_L - \underline{a}_L, \bar{a}_R - \underline{a}_R).$$

See Figure 9 for an illustration. Note that $E \subset \{\Delta_{(0,1]}^Z = a\}$ and if $\Delta_{(0,1]}^Z = a$ and $t_L \leq t_R$, then the event E occurs for some choice of $\mathbf{t}_L, \mathbf{t}_R$, and r . By symmetry between L and R it suffices to bound the regular conditional expectation of $\text{diam}(\mathcal{G}^{2^{-n}}|_{(0,1]})$ given E .

If we condition on E , then the regular conditional law of $Z|_{[0, t_L]}$ is that of a Brownian bridge from 0 to $(-\underline{a}_L, r_L)$ in time t_L conditioned to stay in $[-\underline{a}_L, \infty) \times [-\underline{a}_R, \infty)$; the regular conditional law of $Z|_{[t_L, t_R]}$ is that of a Brownian bridge from $(-\underline{a}_L, r_L)$ to $(r_R, -\underline{a}_R)$ in time $t_R - t_L$ conditioned to stay in $[-\underline{a}_L, \infty) \times [-\underline{a}_R, \infty)$; and the regular conditional law of $Z|_{[t_R, 1]}$ is that of a Brownian bridge from $(r_R, -\underline{a}_R)$ to $(\bar{a}_L - \underline{a}_L, \bar{a}_R - \underline{a}_R)$ in time $t_R - t_L$ conditioned to stay in $[-\underline{a}_L, \infty) \times [-\underline{a}_R, \infty)$.

The diameter of $\mathcal{G}^{2^{-n}}|_{(0,1]}$ is at most the sum of the diameters of its restrictions to $(0, t_L]$, $(t_L, t_R]$, and $(t_R, 1]$. If any of these intervals has length $\leq 2^{-n}$, then the corresponding restriction has diameter either 0 or 1. On the other hand, if the event F_n of Lemma 2.6 occurs and $t_L \geq 2^{-n}$, then it must be the case that

$$\sup_{s_1, s_2 \in [0, t_L]} |Z_{s_1} - Z_{s_2}| \leq n t_L^{1/2}.$$

Similar statements hold for $(t_L, t_R]$ and $(t_R, 1]$. Hence if F_n occurs and we re-scale one of these three intervals whose length is at least 2^{-n} to have unit length, the event \tilde{F}_C of (4.11) occurs with $C = n$ and the restriction of Z to this interval (appropriately re-scaled) in place of \tilde{Z} . By Lemma 4.3 applied in each of the three intervals, we obtain

$$\mathbb{E} \left[\text{diam}(\mathcal{G}^{2^{-n}}|_{(0,1]}) \mathbb{1}_{F_n} \mid E \right] \leq n^5 \mathbb{E} \left[\text{diam}(\mathcal{G}^{2^{-n}}|_{(0,1]}) \right]. \quad (4.17)$$

If we average (4.17) over all choices of $\mathbf{t}_L, \mathbf{t}_R$, and r , we obtain (4.1). \square

5 Existence of an exponent via subadditivity

In this section we will prove the existence of the exponent χ in Theorem 1.12 when ϵ is restricted to powers of 2. We will also prove a concentration estimate which says that the diameter of $\mathcal{G}^{2^{-n}}|_{(0,1]}$ is very unlikely to be too much larger than its expected value, which will be used to prove Theorem 1.15. Throughout this section, we fix $\gamma \in (0, 2)$ and for $n \in \mathbb{N}$ we write

$$D_n := \text{diam}(\mathcal{G}^{2^{-n}}|_{(0,1]}). \quad (5.1)$$

The first main result of this section is a version of Theorem 1.12 with ϵ restricted to powers of 2, which will be proven via a subadditivity argument.

Proposition 5.1. *The limit*

$$\chi := \lim_{n \rightarrow \infty} \frac{\log_2 \mathbb{E}[D_n]}{n}$$

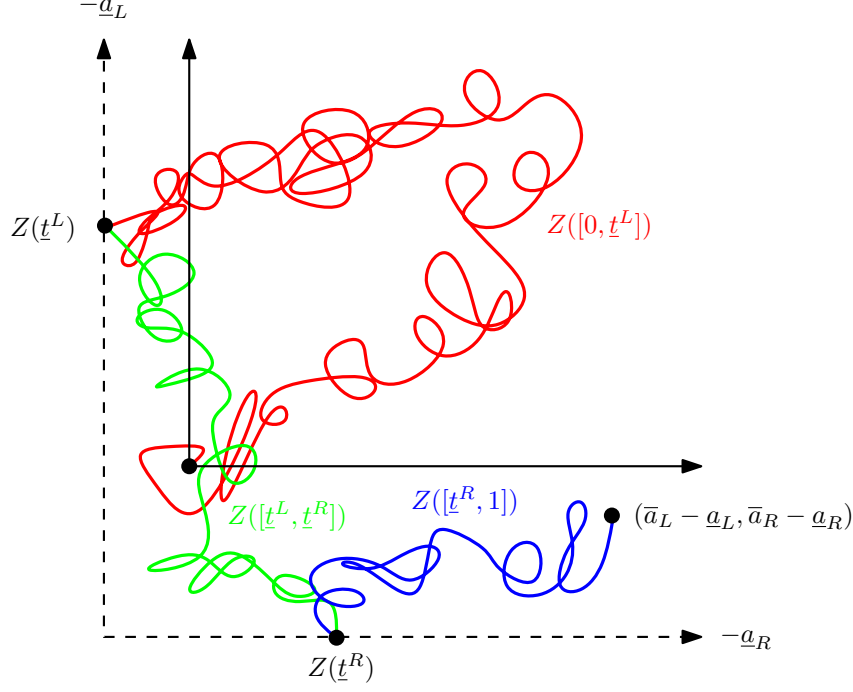


Figure 9: The path $Z|_{[0,1]}$ conditioned on $\{\Delta_{[0,1]}^Z = a\}$, decomposed into three segments as in the proof of Proposition 4.1. If we condition on the event $\{\Delta_{[0,1]}^Z = a\}$ as well as the times \underline{t}^L and \underline{t}^R which separate the three segments and the values of $Z(\underline{t}^L)$ and $Z(\underline{t}^R)$ then the conditional law of each segment is a conditioned Brownian motion to which Lemma 4.3 applies.

exists and satisfies

$$\chi \geq \xi_- \vee \left(1 - \frac{2}{\gamma^2}\right),$$

with ξ_- as in (1.8).

Our other main result is a concentration inequality which says that D_n is at most $2^{(\chi+o_n(1))n}$ with overwhelming probability.

Proposition 5.2. *Let χ be as in Proposition 5.1. There is a constant $c > 0$ depending only on γ such that for each $u \in (0, 1)$ and each $n \in \mathbb{N}$, we have*

$$\mathbb{P}\left[D_n > 2^{(\chi+u)n}\right] \leq \exp(-cu^2n^2) \quad (5.2)$$

with the implicit constant depending only on u and γ .

To prove Proposition 5.1, we start in Section 5.1 by proving a variant of Fekete's subadditivity lemma for a sequence of non-negative real numbers $\{a_n\}_{n \in \mathbb{N}}$ where the subadditivity relation is only required to hold for $m \leq \lambda n$ (for $\lambda \in (0, 1)$ a fixed constant) but a_n is required to be sub-linear. In Section 5.2, we prove a concentration estimate which says that for $m, n \in \mathbb{N}$ with m sufficiently small relative to n , the distance between any two vertices of $\mathcal{G}^{2^{-n-m}}|_{(0,1]}$ is unlikely to differ too much from its conditional expectation given $\{\Delta_{[x-2^{-n}, x]}^Z : x \in (0, 1]_{2^{-n}\mathbb{Z}}\}$ (which determines $\mathcal{G}^{2^{-n}}|_{(0,1]}$). This lemma implies in particular that we can choose a pair of vertices of $\mathcal{G}^{2^{-n-m}}|_{(0,1]}$ whose distance is likely to be close to D_{n+m} in a manner which is measurable with respect to $\{\Delta_{[x-2^{-n}, x]}^Z : x \in (0, 1]_{2^{-n}\mathbb{Z}}\}$. In Section 5.3, we will show (using the estimate of Section 5.2) that the sequence $a_n = \log_2 \mathbb{E}[D_n]$ satisfies the hypotheses of the subadditivity lemma of Section 5.1, and thereby prove Proposition 5.1. In Section 5.4 we will deduce Proposition 5.2 from Proposition 5.1 and the estimate of Section 5.2.

5.1 A variant of Fekete's subadditivity lemma

In this subsection we will prove a variant of Fekete's subadditivity lemma which is one of the main inputs in the proof of Proposition 5.1.

Lemma 5.3. *Fix $\lambda \in (0, 1)$, $C > 0$, and $p \in (0, 1)$. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers which satisfies the restricted subadditivity condition*

$$a_{n+m} \leq a_n + a_m + Cn^p, \quad \forall n, m \in \mathbb{N} \text{ with } n^p \leq m \leq \lambda n \quad (5.3)$$

plus the additional condition

$$a_n \leq Cn, \quad \forall n \in \mathbb{N}. \quad (5.4)$$

Then the limit $\lim_{n \rightarrow \infty} a_n/n$ exists and is finite.

The main point of Lemma 5.3 is that the subadditivity relation 5.3 is only required to hold for $n^p \leq m \leq \lambda n$. Without this restriction, the lemma is an easy consequence of Fekete's lemma and its generalization due to de Bruijn-Erdős [dBE52] (even without the hypothesis (5.4)).

The following recursive relation is the key observation for the proof of Lemma 5.3.

Lemma 5.4. *Assume we are in the setting of Lemma 5.3. For each $n, m \in \mathbb{N}$ with $n^p \leq m \leq \lambda n$, we have*

$$a_n \leq \frac{n}{m} a_m + C(\lambda^{-1} + 1)m + C \frac{n^{1+p}}{m}.$$

Proof. Let $k_* := \lfloor n/m - \lambda^{-1} \rfloor$ be the largest $k \in \mathbb{N}$ for which $n - km \geq \lambda^{-1}m$. Note that $k_* \leq n/m - \lambda^{-1}$. By the subadditivity hypothesis (5.3), for each $k \in [0, k_*]_{\mathbb{Z}}$ we have

$$a_{n-(k-1)m} \leq a_m + a_{n-km} + C(n - km)^p.$$

By iterating this estimate k_* times we get

$$a_n \leq k_* a_m + a_{n-k_*m} + Ck_* n^p. \quad (5.5)$$

We have $k_* \leq n/m$ and by maximality of k_* we have $n - k_*m \leq (\lambda^{-1} + 1)m$ so our sub-linearity hypothesis (5.4) implies $a_{n-k_*m} \leq C(\lambda^{-1} + 1)m$. Thus the statement of the lemma follows from (5.5). \square

Lemma 5.5. *Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be non-decreasing functions and suppose there exists $n_0 \in \mathbb{N}$ such that $f(n) > n$ and $g(n) \geq f(f(n))$ for $n \geq n_0$. Let $\{b_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers and suppose there exists a $\chi > 0$ with the following property. For each sequence $\{n_k\}_{k \in \mathbb{N}}$ with $n_k \rightarrow \infty$ and $f(n_k) \leq n_{k+1} \leq g(n_k)$ for each $k \in \mathbb{N}$, we have $\lim_{k \rightarrow \infty} b_{n_k} = \chi$. Then $\lim_{n \rightarrow \infty} b_n = \chi$.*

Proof. For $r \in \mathbb{N}$, let f^r and g^r be the r -fold compositions of f and g , respectively.

Suppose that $\{m_j\}_{j \in \mathbb{N}}$ is an increasing sequence of positive integers with $m_1 \geq n_0$ and $m_{j+1} \geq g(m_j)$ for each $j \in \mathbb{N}$. We claim that $\lim_{j \rightarrow \infty} b_{m_j} = \chi$. To see this, we will construct a sequence $\{n_k\}_{k \in \mathbb{N}}$ with $f(n_k) \leq n_{k+1} \leq g(n_k)$ for each $k \in \mathbb{N}$ such that $\{m_j\}_{j \in \mathbb{N}}$ is a subsequence of $\{n_k\}_{k \in \mathbb{N}}$. Let $r_1 = 1$ and for $j \geq 2$, let r_j be chosen so that $f^{r_j}(m_{j-1}) \leq m_j < f^{r_j+1}(m_{j-1})$. Such an r_j exists since $m_{j-1} \geq n_0$ so $f^r(m_{j-1}) \geq f^{r-1}(m_{j-1}) + 1$ for $r \in \mathbb{N}$, whence $\lim_{r \rightarrow \infty} f^r(m_{j-1}) = \infty$.

Since $m_j \geq g(m_{j-1}) \geq f^2(m_{j-1})$ we have $r_j \geq 2$. Therefore $f^{r_j-1}(m_{j-1}) \geq m_{j-1} \geq n_0$. By definition of r_j and $g(n) \geq f(f(n))$ for $n \geq n_0$,

$$g(f^{r_j-1}(m_{j-1})) \geq f^{r_j+1}(m_{j-1}) \geq m_j. \quad (5.6)$$

For $j \in \mathbb{N}$, let $k_j := \sum_{i=1}^j r_i$. Let $n_1 := m_1$. For $j \geq 2$ and $k \in (k_{j-1}, k_j)_{\mathbb{Z}}$, let $n_k := f^{k-k_{j-1}}(m_{j-1})$. Let $n_{k_j} := m_j$. We claim that $f(n_k) \leq n_{k+1} \leq g(n_k)$ for each $k \in \mathbb{N}$. Indeed, given $k \in \mathbb{N}$, let $j \in \mathbb{N}$ be chosen so that $k \in [k_{j-1}, k_j - 1]_{\mathbb{Z}}$. If $k \neq k_j - 1$, then we have $n_{k+1} = f(n_k)$, so clearly the desired inequalities hold in this case. If $k = k_j - 1$, then we have $n_{k+1} = m_j$ and $n_k = f^{k_j-k_{j-1}-1}(m_{j-1}) = f^{r_j-1}(m_{j-1})$. By (5.6) we have $g(n_k) \geq n_{k+1}$ and by definition of r_j we have $f(n_k) \leq n_{k+1}$, as required. Since $\lim_{k \rightarrow \infty} b_{n_k} = \chi$ (by hypothesis) we also have $\lim_{j \rightarrow \infty} b_{m_j} = \chi$.

We now argue that $\lim_{n \rightarrow \infty} b_n = \chi$. If not, we can find an increasing sequence $m_j \rightarrow \infty$ and an $\epsilon > 0$ such that for each $j \in \mathbb{N}$ we have $|b_{m_j} - \chi| \geq \epsilon$. By passing to a subsequence we can arrange that $m_1 \geq n_0$ and $m_{j+1} \geq g(m_j)$ for each $j \in \mathbb{N}$. Then the claim above implies that $\lim_{j \rightarrow \infty} b_{m_j} = \chi$, which is a contradiction. \square

Proof of Proposition 5.3. Fix $q \in (1, p^{-1/4})$ and $\hat{q} \in (q^2, p^{-1/2})$. For $n \in \mathbb{N}$ let $f(n) := \lceil n^q \rceil$ and $g(n) := \lfloor n^{\hat{q}} \rfloor$. Observe that f and g satisfy the hypotheses of Lemma 5.5. By Lemma 5.5 it suffices to show that there is a $\chi \in \mathbb{R}$ such that for each sequence $\{n_k\}_{k \in \mathbb{N}}$ with $n_1 \geq 2$ and $n_k^q \leq n_{k+1} \leq n_k^{\hat{q}}$ for each $k \in \mathbb{N}$, we have $\lim_{k \rightarrow \infty} a_{n_k}/n_k = \chi$.

Fix such a sequence $\{n_k\}_{k \in \mathbb{N}}$ and let $b_k := a_{n_k}/n_k$. Since $\hat{q} < p^{-1}$, there is a $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, we have $n_{k+1}^p \leq n_k \leq \lambda n_{k+1}$. By Lemma 5.4, for $k \geq k_0$ we have

$$a_{n_{k+1}} \leq \frac{n_{k+1}}{n_k} a_{n_k} + C(\lambda^{-1} + 1)n_k + C \frac{n_{k+1}^{p+1}}{n_k}$$

Dividing by n_{k+1} gives

$$b_{k+1} \leq b_k + u_k, \quad \text{where} \quad u_k := C(\lambda^{-1} + 1) \frac{n_k}{n_{k+1}} + \frac{n_{k+1}^p}{n_k}. \quad (5.7)$$

Since $n_1 \geq 2$ and $n_k^q \leq n_{k+1} \leq n_k^{\hat{q}}$ for each $k \in \mathbb{N}$ we have $n_k \geq 2^{q^{k-1}}$ for each $k \in \mathbb{N}$ and

$$u_k \leq C(\lambda^{-1} + 1)n_k^{-(q-1)} + n_k^{-(1-\hat{q}p)} \leq O_k(1) \left(2^{-(q-1)q^{k-1}} + 2^{-(1-\hat{q}p)q^{k-1}} \right).$$

Since $1 < q < \hat{q} < p^{-1/2}$, this is summable. Let

$$\tilde{b}_k := b_k - \sum_{j=1}^{k-1} u_j \quad \text{and} \quad \beta := \sum_{j=1}^{\infty} u_j.$$

The relation (5.7) implies that $\tilde{b}_{k+1} \leq \tilde{b}_k$ for each $k \in \mathbb{N}$. Since $\tilde{b}_k \geq -\beta$ for each k , we infer that $\lim_{k \rightarrow \infty} \tilde{b}_k$ exists. Hence also

$$\chi := \lim_{k \rightarrow \infty} b_k = \lim_{k \rightarrow \infty} \tilde{b}_k + \beta$$

exists. It remains to show that the χ does not depend on the initial choice of sequence $\{n_k\}_{k \in \mathbb{N}}$. To this end, it is enough to show that

$$\limsup_{n \rightarrow \infty} \frac{a_n}{n} \leq \chi, \quad (5.8)$$

since then the limiting values χ arising from two different choices of subsequence must agree by symmetry.

To prove (5.8), suppose given $n \in \mathbb{N}$ with $n \geq n_{k_0+2}$ (with k_0 defined as in the beginning of the proof). Let $k \in \mathbb{N}$ be the largest integer such that $n_{k+1} \leq n$, and note that $k \geq k_0$. Then our condition on the n_k 's implies that $n^{1/\hat{q}^2} \leq n_k \leq n^{1/q}$. Since $1/q < 1$ and $1/\hat{q}^2 > p$, Lemma 5.4 with $m = n_k$ implies that

$$\begin{aligned} a_n &\leq \frac{n}{n_k} a_{n_k} + C(\lambda^{-1} + 1)n_k + C \frac{n^{1+p}}{n_k} \\ &\leq \chi n(1 + o_n(1)) + C(\lambda^{-1} + 1)n^{1/q} + Cn^{1+p-1/\hat{q}^2} \\ &= \chi n(1 + o_n(1)). \end{aligned}$$

□

5.2 Conditioned concentration bound

For $n \in \mathbb{N}$, let

$$\mathcal{H}^n := \sigma\left(\Delta_{[x-2^{-n}, x]}^Z : x \in (0, 1]_{2^{-n}\mathbb{Z}}\right), \quad (5.9)$$

so that, by Lemma 2.3, the graph $\mathcal{G}^{2^{-n}}|_{(0,1]}$ is \mathcal{H}^n -measurable. In this subsection we will prove the following concentration bound, which says that distances in $\mathcal{G}^{2^{-n-m}}|_{(0,1]}$ are unlikely to differ very much from their expected values given \mathcal{H}^n .

Proposition 5.6. *Let $n, m \in \mathbb{N}$ and let \mathcal{H}^n be the σ -algebra defined in (5.9). Let $y_0, y_1 \in (0, 1]_{2^{-n-m}\mathbb{Z}}$ be chosen in a \mathcal{H}^n -measurable manner. Then for $t > 0$ we have*

$$\mathbb{P}\left[\left|\text{dist}\left(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]}\right) - \mathbb{E}\left[\text{dist}\left(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]}\right) \mid \mathcal{H}^n\right]\right| > t \mid \mathcal{H}^n\right] \leq 2 \exp\left(-\frac{t^2}{2^{3m+1}D_n}\right). \quad (5.10)$$

Remark 5.7. Proposition 5.6 is needed for the proofs of both Proposition 5.1 and 5.2. The relevance to Proposition 5.2 is clear. The reason we need the conditional concentration bound for the proof of Proposition 5.1 is more subtle. Let $n, m \in \mathbb{N}$ with m at most a small constant times n . Suppose we condition on \mathcal{H}^n and choose vertices x_0, x_1 of $\mathcal{G}^{2^{-n}}|_{(0,1]}$ lying at distance D_n from one another. Consider a graph-distance geodesic (of length D_n) from x_0 to x_1 in $\mathcal{G}^{2^{-n}}|_{(0,1]}$, then use Proposition 4.1 to argue that the conditional expected diameter of $\mathcal{G}^{2^{-n-m}}|_{(x-2^{-n}, x]}$ given \mathcal{H}^n for each vertex x along this geodesic is not too much bigger than $\mathbb{E}[D_m]$. Summing over all of these vertices shows that the expected distance between x_0 and x_1 in $\mathcal{G}^{2^{-n-m}}|_{(0,1]}$ is not much bigger than $\mathbb{E}[D_m]\mathbb{E}[D_n]$. This would allow us to apply Lemma 5.3 with $a_n = \log_2 \mathbb{E}[D_n]$ if we could bound $\mathbb{E}[D_{n+m}]$ instead of $\mathbb{E}[\text{dist}(x_0, x_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]})]$. However, it is possible that the pair of vertices in $\mathcal{G}^{2^{-n-m}}|_{(0,1]}$ which lie at maximal distance is very different from the pair of vertices in $\mathcal{G}^{2^{-n}}|_{(0,1]}$ which lie at maximal distance. Proposition 5.6 tells us that we can choose $y_0, y_1 \in (0, 1]_{2^{-n-m}\mathbb{Z}}$ in a \mathcal{H}^n -measurable manner in such a way that $\text{dist}(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]})$ is likely to be close to D_{n+m} (namely, we choose y_0 and y_1 so as to maximize $\mathbb{E}[\text{dist}(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]}) \mid \mathcal{H}^n]$).

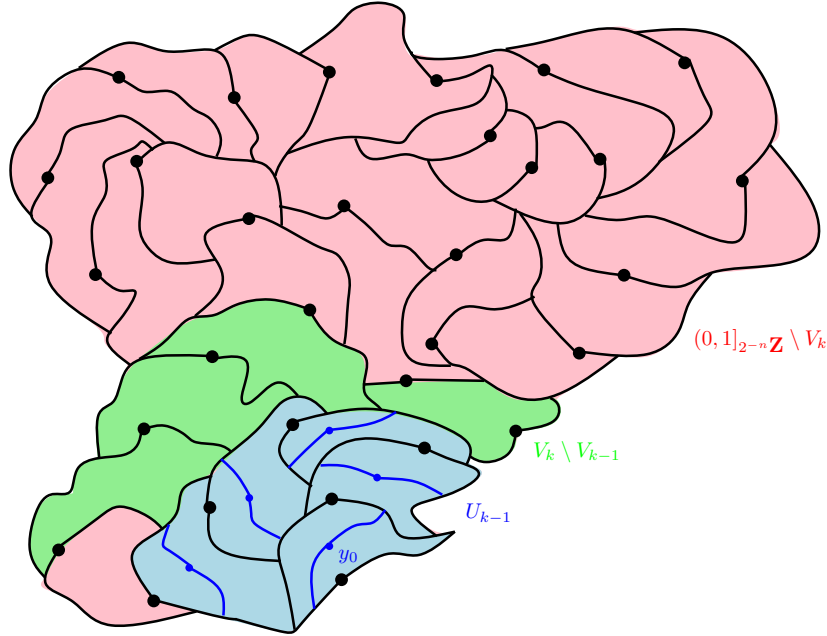


Figure 10: An illustration of the graph G_{k-1} used in the proof of Proposition 5.6 in the case when $m = 1$ (the case for $m \geq 2$ is similar, but cells are subdivided into 2^m , rather than 2, pieces). The cells of $\mathcal{G}^{2^{-n}}|_{(0,1]}$ are shown in black. Elements of U_{k-1} (corresponding to cells in light blue) each correspond to half of one of the cells in $\mathcal{G}^{2^{-n}}|_{(0,1]}$. Elements of $V_k \setminus V_{k-1}$ (indicated by green dots) correspond to cells of $\mathcal{G}^{2^{-n}}|_{(0,1]}$ which have not yet been subdivided, but which lie at minimal distance to y_0 in G_{k-1} among all such cells. Cells of $\mathcal{G}^{2^{-n}}|_{(0,1]}$ which have not yet been subdivided and do not correspond to elements of V_k are shown in pink. The graph G_k is obtained from G_{k-1} by subdividing each of the green cells.

Proposition 5.6 will eventually be extracted from Azuma's inequality. To this end, we first construct a sequence of graphs which interpolate between $\mathcal{G}^{2^{-n}}$ and $\mathcal{G}^{2^{-n-m}}$. See Figure 10 for an illustration.

Fix $n, m \in \mathbb{N}$ and $y_0, y_1 \in (0, 1]_{2^{-n}\mathbb{Z}}$ as in the statement of Proposition 5.6. Let $x_0, x_1 \in (0, 1]_{2^{-n}\mathbb{Z}}$ be chosen so that $y_0 \in (x_0 - 2^{-n}, x_0]_{2^{-n-m}\mathbb{Z}}$ and $y_1 \in (x_1 - 2^{-n}, x_1]_{2^{-n-m}\mathbb{Z}}$.

Let $G_0 := \mathcal{G}^{2^{-n}}|_{(0,1]}$ and $V_1 := \{x_0\}$. Inductively, suppose $k \in \mathbb{N}$ and a graph G_{k-1} as well as a set $V_k \subset (0, 1]_{2^{-n}\mathbb{Z}}$ have been defined. Let

$$U_k := \{x - j2^{-m-n} : x \in V_k, j \in [0, 2^m - 1]_{\mathbb{Z}}\}. \quad (5.11)$$

Let G_k be the graph whose vertex set is $((0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k) \sqcup U_k$ with adjacency defined as follows. If $y, y' \in (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k$ (resp. $y, y' \in U_k$), then y and y' are connected by an edge in G_k if and only if they are connected by an edge in $\mathcal{G}^{2^{-n}}|_{(0,1]}$ (resp. $\mathcal{G}^{2^{-n-m}}|_{(0,1]}$). If $y \in (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k$ and $y' \in U_k$, then y and y' are connected by an edge in G_k if and only if the cells $\eta([y - 2^{-n}, y])$ and $\eta([y' - 2^{-n-m}, y'])$ share a non-trivial boundary arc. That is, G_k is a hybrid of $\mathcal{G}^{2^{-n}}|_{(0,1]}$ and $\mathcal{G}^{2^{-n-m}}|_{(0,1]}$ where elements of $(0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k$ correspond to intervals of length 2^{-n} and elements of U_k correspond to intervals of length 2^{-n-m} . Let

$$V_{k+1} := V_k \cup \{x \in (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k : \text{dist}(y_0, x; G_k) = \text{dist}(y_0, (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k; G_k)\}. \quad (5.12)$$

Let $\mathcal{H}_0^n := \mathcal{H}^n$, as in (5.9). For $x \in (0, 1]_{2^{-n}\mathbb{Z}}$, let

$$A_x := \left\{ \Delta_{[x-(j+1)2^{-m-n}, x-j2^{-m-n}]}^Z : x \in V_k, j \in [0, 2^m - 1]_{\mathbb{Z}} \right\} \quad (5.13)$$

so that the random 2^{m+2} -tuples A_x are conditionally independent given \mathcal{H}_0^n and together determine all of the graphs G_k (recall Lemma 2.3). Also let

$$\mathcal{H}_k^n := \mathcal{H}_0^n \vee \sigma(A_x : x \in V_k),$$

so that G_k and V_{k+1} are \mathcal{H}_k^n -measurable.

Let

$$K := \inf \left\{ k \in \mathbb{N} : G_k = \mathcal{G}^{2^{-n-m}}|_{(0,1]} \right\}.$$

Note that $G_k = G_K = \mathcal{G}^{2^{-n-m}}|_{(0,1]_{\mathbb{Z}}}$ for each $k \geq K$. The graph distance from y_0 to $(0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k$ in G_k increases by at least 1 whenever k increases by 1. Consequently,

$$K \leq D_{n+m} \leq 2^m D_n. \quad (5.14)$$

For $k \geq 0$, let

$$M_k := \mathbb{E} \left[\text{dist}(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]}) \mid \mathcal{H}_k^n \right]. \quad (5.15)$$

Then M is a (\mathcal{H}_k^n) -martingale (n fixed) and $M_k = \text{dist}(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]})$ for $k \geq K$. The following lemma is the main ingredient in the proof of Proposition 5.6.

Lemma 5.8. *For each $k \in \mathbb{N}$, we have $M_k - M_{k-1} \leq 2^m$.*

For the proof of Lemma 5.8, we recall that a *realization* of a random variable X is an element of the support of the law of X . A realization of a σ -algebra is a realization of a set of random variables which generate it.

The idea of the proof of Lemma 5.8 is as follows. Suppose given $k \in \mathbb{N}$ and condition on realizations of \mathcal{H}_{k-1}^n and $\{A_x : x \in (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k\}$. If we are given two different realizations of $\{A_x : x \in V_k \setminus V_{k-1}\}$ (which correspond to two different ways to subdivide the cells corresponding to elements of $V_k \setminus V_{k-1}$ into 2^{-m} pieces) we obtain two different possible realizations of G_K and hence two different realizations of $\text{dist}(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]})$. We will argue that these two realizations differ by at most 2^m . Averaging over all realizations of $\{A_x : x \in (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k\}$ will show that changing the information in \mathcal{H}_k^n while leaving the information in \mathcal{H}_{k-1}^n fixed can change the value of M_k by at most 2^m , which will imply the statement of the lemma. We now proceed with the details.

Proof of Lemma 5.8. Let $k \in \mathbb{N}$. Throughout the proof we assume that we have conditioned on a realization of \mathcal{H}_{k-1}^n , which determines realizations of \mathcal{H}^n and of V_k and V_{k-1} . Let \mathfrak{A}^1 and \mathfrak{A}^2 be two realizations of

$$\{A_x : x \in V_k \setminus V_{k-1}\} \quad (5.16)$$

which are compatible with our given realizations of \mathcal{H}_{k-1}^n . Note that \mathcal{H}_k^n is generated by \mathcal{H}_{k-1}^n and the random vectors (5.16), so \mathfrak{A}^1 and \mathfrak{A}^2 together with our given realization of \mathcal{H}_{k-1}^n determine two possible realizations of \mathcal{H}_k^n .

Let \mathfrak{X} be a realization of

$$\{A_x : x \in (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k\} \quad (5.17)$$

which is compatible with our given realizations of \mathcal{H}_{k-1}^n . The σ -algebra $\mathcal{H}^{n+m} = \mathcal{H}_K^n$ is generated by \mathcal{H}_{k-1}^n and the random vectors (5.16) and (5.17). For $i \in \{1, 2\}$, let $\mathfrak{G}^i = \mathfrak{G}(\mathfrak{A}^i, \mathfrak{X})$ be the realization of $G_K = \mathcal{G}^{2^{-n-m}}|_{(0,1]}$ which is determined by \mathfrak{A}^i , \mathfrak{X} , and our given realization of \mathcal{H}_{k-1}^n . Also let $\mathfrak{D}(\mathfrak{A}^i, \mathfrak{X})$ be the corresponding realization of $\text{dist}(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]})$.

The random vectors (5.17) are conditionally independent from \mathcal{H}_k^n given \mathcal{H}_{k-1}^n . Consequently, on the event

$$\{\{A_x : x \in V_k \setminus V_{k-1}\} = \mathfrak{A}^i\}$$

for $i \in \{1, 2\}$, the quantity M_k is obtained by integrating $\mathfrak{D}(\mathfrak{A}^i, \mathfrak{X})$ over all possible realizations \mathfrak{X} as above with respect to the conditional law of $\{A_x : x \in (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_k\}$ given \mathcal{H}_{k-1}^n . Furthermore, this conditional law does not depend on \mathfrak{A}^i . Therefore, to prove the lemma, it suffices to show that

$$|\mathfrak{D}(\mathfrak{A}^1, \mathfrak{X}) - \mathfrak{D}(\mathfrak{A}^2, \mathfrak{X})| \leq 2^m. \quad (5.18)$$

The proof of (5.18) is entirely deterministic. See Figure 11 for an illustration. Let $P^1 : [0, |P^1|] \rightarrow \mathcal{V}(\mathfrak{G}^1) = (0, 1]_{2^{-n-m}\mathbb{Z}}$ be a path in \mathfrak{G}^1 from y_0 to y_1 (Definition 1.6). We will construct a path P^2 in \mathfrak{G}^2 from y_0 to y_1 whose length is at most $|P^1| + 2^m$. If P^1 does not pass through $U_k \setminus U_{k-1}$ (defined as in (5.11)), then since the restrictions of \mathfrak{G}^1 and \mathfrak{G}^2 to $(0, 1]_{2^{-n-m}\mathbb{Z}} \setminus (U_k \setminus U_{k-1})$ agree, we can just take $P^2 = P^1$. Hence we can assume without loss of generality that P^1 passes through $U_k \setminus U_{k-1}$. Let

$$\iota := \sup\{i \in [0, |P^1|]_{\mathbb{Z}} : P^1(i) \in U_k \setminus U_{k-1}\}.$$

Then either $\iota = |P^1|$ or the cell $\eta([P^1(\iota + 1) - 2^{-n-m}, P^1(\iota + 1)])$ shares a non-trivial boundary arc with $\eta([x - 2^{-n}, x])$ for some $x \in U_k$. In the former case, we set $\tilde{y} = P^1(\iota) = y_1$. In the latter case, we can choose $\tilde{y} \in U_k$ which is adjacent to $P(\iota + 1)$ in \mathfrak{G}^2 .

Let $\tilde{x} \in V_k$ be chosen so that $\tilde{y} \in (\tilde{x} - 2^{-n}, \tilde{x}]_{2^{-n-m}\mathbb{Z}}$. By the definition (5.12) of V_k and since the restrictions of \mathfrak{G}^2 and G_{k-1} to U_{k-1} agree, we can choose $\tilde{y}' \in (\tilde{x} - 2^{-n}, \tilde{x}]_{2^{-n-m}\mathbb{Z}}$ such that

$$\text{dist}(y_0, \tilde{y}'; \mathfrak{G}^2) = \text{dist}(y_0, (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_{k-1}; \mathfrak{G}^2) = \text{dist}(y_0, (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_{k-1}; G_{k-1}).$$

Since the restrictions of \mathfrak{G}^1 and G_{k-1} to U_{k-1} agree, we have

$$\text{dist}(y_0, \tilde{y}'; \mathfrak{G}^2) = \text{dist}(y_0, (0, 1]_{2^{-n}\mathbb{Z}} \setminus V_{k-1}; \mathfrak{G}^1) \leq \iota.$$

Since \tilde{y} and \tilde{y}' both belong to $(\tilde{x} - 2^{-n}, \tilde{x}]_{2^{-n-m}\mathbb{Z}}$, it follows that \tilde{y} lies at \mathfrak{G}^2 -graph distance at most 2^m from \tilde{y}' . Consequently, we can find a path \tilde{P}^2 from y_0 to \tilde{y} in \mathfrak{G}^2 of length at most $\iota + 2^m$. Let P^2 be the concatenation of \tilde{P}^2 and $P^1|_{[\iota+1, |P^1|]}$. Since $P^1|_{[\iota+1, |P^1|]}$ does not contain any vertex in $U_k \setminus U_{k-1}$, it follows that P^2 is a path from y_0 to y_1 in \mathfrak{G}^2 of length at most $|P^1| + 2^m$. By symmetry of \mathfrak{G}^1 and \mathfrak{G}^2 , we infer that (5.18) holds. \square

Proof of Proposition 5.6. By Lemma 5.8 and Azuma's inequality, we infer that for $t > 0$,

$$\mathbb{P}[|M_{2^m D_n} - M_0| > t \mid \mathcal{H}_0^n] \leq 2 \exp\left(-\frac{t^2}{2^{3m+1} D_n}\right).$$

In light of (5.14), this implies (5.10). \square

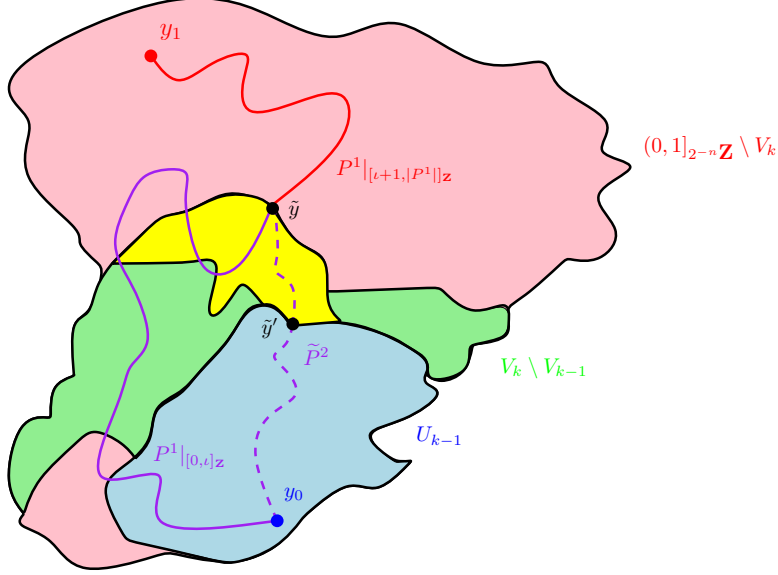


Figure 11: An illustration of the proof of Lemma 5.8. Shown is the set $\eta([0, 1])$, without all cell subdivisions shown explicitly. The union of the cells corresponding to elements of V_{k-1} (resp. $V_k \setminus V_{k-1}$, $(0, 1]_{2^{-n}} \setminus V_k$) is shown in light blue (resp. light green and yellow, pink). The graphs \mathfrak{G}^1 and \mathfrak{G}^2 agree except with regard to how the cells in the light green region are subdivided. Given a path P^1 from y_0 to y_1 in \mathfrak{G}^1 (solid purple and red lines) we consider the last time ι that P^1 exits $V_k \setminus V_{k-1}$. By the definition (5.12) of V_k and since the cell of $\mathcal{G}^{2^{-n}}$ containing $P^1(\iota)$ (shown in yellow) contains 2^m cells of $\mathcal{G}^{2^{-n-m}}$, we can find a path \tilde{P}^2 from y_0 to $P^1(\iota + 1)$ in \mathfrak{G}^2 with length at most $\iota + 2^m$ (dotted purple line). We then concatenate this path with the part of P^1 traced after time ι (red line).

5.3 Proof Proposition 5.1

In this subsection we will deduce Proposition 5.1 by checking the hypotheses of Proposition 5.3 with $a_n = \log_2 \mathbb{E}[D_n]$. Throughout this section and the next, we define the event F_n for $n \in \mathbb{N}$ as in Lemma 2.6. For $x, y \in \mathbb{R}$ with $x < y$, we let $F_n(x, y)$ be the event that F_n occurs with the re-scaled Brownian motion $t \mapsto (y - x)^{-1/2}(Z_{t(y-x)+x} - Z_x)$ in place of Z . We set

$$\hat{F}_{n,m} := \bigcap_{x \in (0,1]_{2^{-n}\mathbb{Z}}} F_m(x - 2^{-n}, x). \quad (5.19)$$

By Lemma 2.6 and the union bound, we have

$$\mathbb{P}[\hat{F}_{n,m}^c] \leq c_0 e^{-c_1 m^2 + n} \quad (5.20)$$

for universal constants $c_0, c_1 > 0$.

Lemma 5.9. *Let $n, m \in \mathbb{N}$. Let \mathcal{H}^n be as in (5.9) and let $\hat{F}_{n,m}$ be as in (5.19). Let $y_0, y_1 \in (0, 1]_{2^{-n-m}\mathbb{Z}}$ be chosen in a \mathcal{H}^n -measurable manner. Also let $x_0, x_1 \in (0, 1]_{2^{-n}\mathbb{Z}}$ be chosen so that $y_0 \in (x_0 - 2^{-n}, x_0]$ and $y_1 \in (x_1 - 2^{-n}, x_1]$. Then*

$$\mathbb{E}[\text{dist}(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]}) \mathbb{1}_{\hat{F}_{n,m}} | \mathcal{H}^n] \preceq n^5 \mathbb{E}[D_m] \text{dist}(x_0, x_1; \mathcal{G}^{2^{-n}}|_{(0,1]}),$$

with the implicit constant depending only on γ .

Proof. Let $P : [1, |P|]_{\mathbb{Z}} \rightarrow (0, 1]_{2^{-n}\mathbb{Z}}$ be a path in $\mathcal{G}^{2^{-n}}|_{(0,1]}$ from x_0 to x_1 with $|P| = \text{dist}(x_0, x_1; \mathcal{G}^{2^{-n}}|_{(0,1]})$,

chosen in some \mathcal{H}^n -measurable manner. Then

$$\text{dist}\left(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]}\right) \mathbb{1}_{\widehat{F}_{n,m}} \leq \sum_{i=1}^{|P|} \text{diam}\left(\mathcal{G}^{2^{-n-m}}|_{(P(i)-2^{-n}, P(i)]}\right) \mathbb{1}_{F_m(P(i)-2^{-n}, P(i))}. \quad (5.21)$$

The conditional law given \mathcal{H}^n of each of the restricted Brownian motions $Z|_{(P(i)-2^{-n}, P(i)]}$ is the same as its conditional law given $\Delta_{(P(i)-2^{-n}, P(i))}^Z$. By Proposition 4.1 and scale invariance, we find that the conditional expectation given \mathcal{H}^n of each term in the sum on the right in (5.21) is $\preceq n^5 \mathbb{E}[D_m]$. \square

We next transfer from the distance estimate of Lemma 5.9 to a diameter estimate using Proposition 5.6.

Lemma 5.10. *Suppose $n, m \in \mathbb{N}$ and $\zeta > 0$ with $m \leq 2^{\zeta n}$. There are constants $b_0, b_1 > 0$, depending only on ζ and γ , such that the following is true. We have*

$$\mathbb{P}\left[D_{n+m} > b_1 n^5 \mathbb{E}[D_m] D_n + 2^{\zeta n+3m/2} D_n^{1/2}, \widehat{F}_{n,m} \mid \mathcal{H}^n\right] \preceq \exp(-b_0 2^{2\zeta n}) \quad (5.22)$$

and

$$\mathbb{E}\left[D_{n+m} \mathbb{1}_{\widehat{F}_{n,m}} \mid \mathcal{H}^n\right] \preceq n^5 \mathbb{E}[D_m] D_n + 2^{\zeta n+3m/2} D_n^{1/2} \quad (5.23)$$

with deterministic implicit constants depending only on ζ and γ .

Proof. Proposition 5.6 and a union bound imply that there is a constant $b_0 > 0$ depending only on γ and ζ such that the following is true. Except on an event of conditional probability $\preceq \exp(-b_0 2^{2\zeta n})$ given \mathcal{H}^n , we have

$$\sup_{y_0, y_1 \in (0,1]_{2^{-n-m}\mathbb{Z}}} \left| \text{dist}\left(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]}\right) - \mathbb{E}\left[\text{dist}\left(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]}\right) \mid \mathcal{H}^n\right] \right| \leq 2^{\zeta n+3m/2} D_n^{1/2}.$$

By combining this with Lemma 5.9, we find that it holds except on an event of conditional probability $\preceq \exp(-b_0 2^{2\zeta n})$ given \mathcal{H}^n that either $\widehat{F}_{n,m}^c$ occurs or

$$\begin{aligned} D_{n+m} &\leq \sup_{y_0, y_1 \in (0,1]_{2^{-n-m}\mathbb{Z}}} \mathbb{E}\left[\text{dist}\left(y_0, y_1; \mathcal{G}^{2^{-n-m}}|_{(0,1]}\right) \mid \mathcal{H}^n\right] + 2^{\zeta n+3m/2} D_n^{1/2} \\ &\leq b_1 n^5 \mathbb{E}[D_m] D_n + 2^{\zeta n+3m/2} D_n^{1/2} \end{aligned} \quad (5.24)$$

for an appropriate constant $b_1 > 0$ as in the statement of the lemma. This immediately implies (5.22). Since $D_{n+m} \leq 2^{n+m} \preceq \exp(b_0 2^{2\zeta n})$, we can take the conditional expectations given \mathcal{H}^n of both sides of (5.24) to obtain (5.23). \square

The following lemma shows that $\log_2 D_n$ satisfies the restricted subadditivity condition in Lemma 5.3.

Lemma 5.11. *Fix*

$$\lambda \in \left(0, \frac{2}{8 + 6\sqrt{2}\gamma + 3\gamma^2}\right). \quad (5.25)$$

For each $n, m \in \mathbb{N}$ with $n^{2/3} \leq m \leq \lambda n$, we have

$$\mathbb{E}[D_{n+m}] \preceq n^5 \mathbb{E}[D_m] \mathbb{E}[D_n]$$

with implicit constant depending only on λ and γ .

Proof. By taking expectations of both sides of the estimate (5.23) of Lemma 5.10 we obtain that for each $\zeta > 0$,

$$\begin{aligned} \mathbb{E}[D_{n+m}] &\preceq n^5 \mathbb{E}[D_m] \mathbb{E}[D_n] + 2^{\zeta n/2+3m/2} \mathbb{E}\left[D_n^{1/2}\right] + 2^{n+m} \mathbb{P}\left[\widehat{F}_{n,m}^c\right] \\ &\preceq \mathbb{E}[D_n] \mathbb{E}[D_m] \left(n^5 + \frac{2^{\zeta n+3m/2}}{\mathbb{E}[D_n]^{1/2} \mathbb{E}[D_m]} \right). \end{aligned} \quad (5.26)$$

Here we use that $D_{n+m} \leq 2^{n+m} \leq 1/\mathbb{P}[\widehat{F}_{n,m}^c]$ (recall (5.20) and the assumption that $m \geq n^{2/3}$) and we apply Jensen's inequality to bring an exponent of $1/2$ outside of the expectation.

Let $\alpha > 0$ with

$$\alpha < \frac{1}{2 + \gamma^2/2 + \sqrt{2}\gamma}. \quad (5.27)$$

By Proposition 3.1 and (5.26), for each $\zeta > 0$ we have

$$\mathbb{E}[D_{n+m}] \leq \mathbb{E}[D_n]\mathbb{E}[D_m] \left(n^5 + 2^{(\zeta - \alpha/2)n + (3/2 - \alpha)m} \right). \quad (5.28)$$

If $m \leq \lambda n$ and we choose ζ sufficiently small and α sufficiently close to the right side of (5.27) then $(\zeta - \alpha/2)n + (3/2 - \alpha)m < 0$, so the right side of (5.28) is $\leq n^5 \mathbb{E}[D_n]\mathbb{E}[D_m]$. \square

Proof of Proposition 5.1. By Lemma 5.11, we find that the hypotheses of Lemma 5.3 are satisfied for λ as in (5.25), $p = 2/3$, $a_n = \log_2 \mathbb{E}[D_n]$, and some $C \geq 1$. Consequently, Lemma 5.3 implies the existence of the limit defining χ . The lower bound for χ is immediate from Proposition 3.1. \square

5.4 Proof of Proposition 5.2

In this subsection we will deduce Proposition 5.2 from the earlier results of this subsection. We continue to use the notations (5.9) and (5.19).

Proof of Proposition 5.2. Fix $u \in (0, 1)$. Let λ be as in (5.25) and let

$$\zeta \in \left(\frac{u}{8\lambda^{-1}\chi^{-1} + 4}, \frac{u}{4\lambda^{-1}\chi^{-1} + 4} \right).$$

Let $k_* = \lfloor (\lambda\zeta)^{-1} - 1 \rfloor$ be the largest $k \in \mathbb{N}$ for which

$$1 - k\zeta \geq (\lambda^{-1} \vee 4\chi^{-1})\zeta. \quad (5.29)$$

By Proposition 5.1, there exists a function $\phi_0 : [0, \infty) \rightarrow \mathbb{R}$ with $\lim_{t \rightarrow \infty} t^{-\alpha} \phi(t) = 0$ for each $\alpha > 0$ such that

$$\mathbb{E}[D_n] = \phi_0(2^n)2^{\chi n}, \quad \forall n \in \mathbb{N}. \quad (5.30)$$

By Lemma 5.10 we can find constants $b_0, b_1 > 0$ depending only on ζ, λ , and γ such that for each $k \in [1, k_*]_{\mathbb{Z}}$, we have

$$\mathbb{P} \left[D_{n-(k-1)\lfloor \zeta n \rfloor} > D_{n-k\lfloor \zeta n \rfloor} \left(b_1 n^5 \phi_0(2^{\zeta n}) 2^{\chi \zeta n} + 2^{2\zeta n} D_{n-k\lfloor \zeta n \rfloor}^{-1/2} \right), \widehat{F}_{n-k\lfloor \zeta n \rfloor, \lfloor \zeta n \rfloor} \right] \leq \exp \left(-b_0 2^{\zeta(1-k\zeta)n} \right) \quad (5.31)$$

where here $\widehat{F}_{n-k\lfloor \zeta n \rfloor, \lfloor \zeta n \rfloor}$ is as in (5.19). By iterating the estimate (5.31) k_* times we find that the following is true. Let

$$E := \left\{ D_n \leq D_{n-k\lfloor \zeta n \rfloor} \prod_{j=1}^k \left(b_1 n^5 \phi_0(2^{\zeta n}) 2^{\chi \zeta n} + 2^{2\zeta n} D_{n-j\lfloor \zeta n \rfloor}^{-1/2} \right), \forall k \in [1, k_*]_{\mathbb{Z}} \right\}.$$

Then

$$\mathbb{P} \left[E^c \cap \bigcap_{k=1}^{k_*} \widehat{F}_{n-k\lfloor \zeta n \rfloor, \lfloor \zeta n \rfloor} \right] \leq \exp \left(-b_0 2^{\zeta^2 n} \right). \quad (5.32)$$

By (5.20) and the union bound,

$$\mathbb{P} \left[\bigcap_{k=1}^{k_*} \widehat{F}_{n-k\lfloor \zeta n \rfloor, \lfloor \zeta n \rfloor} \right] \leq \exp \left(-cu^2 n^2 \right) \quad (5.33)$$

for $c > 0$ a constant depending only on γ (here we recall that $\zeta \succeq u$). Therefore, $\mathbb{P}[E^c] \preceq \exp(-cu^2n^2)$.

We will complete the proof by showing that if E occurs and n is sufficiently large, then $D_n \leq 2^{(\chi+u)n}$. Suppose to the contrary that E occurs and $D_n > 2^{(\chi+u)n}$. Let k_0 be the smallest $k \in [1, k_* - 1]_{\mathbb{Z}}$ for which $D_{n-k\lfloor\zeta n\rfloor} \leq 2^{\chi(1-k\zeta)n}$ or $k_0 = k_*$ if no such k exists. By definition of E , we have

$$D_n \leq D_{n-k_0\lfloor\zeta n\rfloor} (b_1 n^5 \phi_0(2^{\zeta n}) 2^{\chi\zeta n} + 2^{2\zeta n}) \prod_{j=1}^{k_0-1} (b_1 n^5 \phi_0(2^{\zeta n}) 2^{\chi\zeta n} + 2^{(2\zeta-\chi(1-j\zeta)/2)n}). \quad (5.34)$$

By (5.29),

$$2^{2\zeta-\chi(1-j\zeta)/2} \leq 1, \quad \forall j \in [1, k_*]_{\mathbb{Z}}.$$

By (5.34), we infer that

$$\begin{aligned} D_n &\leq D_{n-k_0\lfloor\zeta n\rfloor} (b_1 n^5 \phi_0(2^{\zeta n}) 2^{\chi\zeta n} + 2^{2\zeta n}) \prod_{j=1}^{k_0-1} (b_1 n^5 \phi_0(2^{\zeta n}) 2^{\chi\zeta n} + 1) \\ &\leq ((b_1 + 1)n)^{5k_*} \phi_0(2^{\zeta n})^{5k_*} 2^{(\chi\zeta k_0 + 2\zeta)n} D_{n-k_0\lfloor\zeta n\rfloor}. \end{aligned} \quad (5.35)$$

We have $((b_1 + 1)n)^{5k_*} \phi_0(2^{\zeta n})^{5k_*} \leq 2^{o(n)}$ (at a rate depending on ζ), so for large enough n ,

$$D_n \leq 2^{(\chi\zeta k_0 + 3\zeta)n} D_{n-k_0\lfloor\zeta n\rfloor}. \quad (5.36)$$

If $k_0 < k_*$, then $D_{n-k_0\lfloor\zeta n\rfloor} \leq 2^{\chi(1-k_0\zeta)n}$ so $D_n \leq 2^{(\chi+3\zeta)n} \leq 2^{(\chi+u)n}$. If $k_0 = k_*$, then $D_{n-k_0\lfloor\zeta n\rfloor} \leq 2^{(1-\zeta k_*)n+1} \leq 2^{(C+1)\zeta n+1}$ for $C = (\lambda^{-1} \vee 4\chi^{-1})$ so by (5.36) and our choice of ζ we have $D_n \leq 2^{(\chi+(C+4)\zeta)n} \leq 2^{(\chi+u)n}$ for large enough n . Hence if n is chosen sufficiently large (depending only on ζ) then on E we have $D_n \leq 2^{(\chi+u)n}$, as required. \square

6 General distance estimates

In this section we will prove some extensions of Propositions 5.1 and 5.2 which will eventually be used to prove Theorems 1.12 and 1.15. Throughout, we let χ be the exponent from Proposition 5.1.

6.1 Expected distance between uniformly random or fixed points

In this subsection we will transfer our diameter estimate Proposition 5.1 to an estimate for expected distances between particular pairs of vertices in $\mathcal{G}^{2^{-n}}|_{(0,1]}$.

Proposition 6.1. *With χ as in Proposition 5.1,*

$$\lim_{n \rightarrow \infty} \frac{\log_2 \mathbb{E}[X_n]}{n} = \chi, \quad (6.1)$$

where X_n is either of the following two random variables.

1. $X_n = \text{dist}(x_0^n, x_1^n; \mathcal{G}^{2^{-n}}|_{(0,1]})$ where x_0^n is chosen in some $\mathcal{G}^{2^{-n}}|_{(0,1]}$ -measurable manner and x_1^n is sampled uniformly from $(0, 1]_{2^{-n}\mathbb{Z}}$, independently from $\mathcal{G}^{2^{-n}}|_{(0,1]}$.
2. $X_n = \text{dist}(2^{-n}, 1; \mathcal{G}^{2^{-n}}|_{(0,1]})$.

Throughout this subsection, we define the diameter D_n as in (5.1). For the proof of Proposition 6.1 we will need several lemmas, which are all straightforward consequences of Propositions 5.1 and 5.2 from the previous section. Our first lemma tells us in particular that the probability that D_n is smaller than its expected value by more than an exponential factor decays slower than any exponential function.

Lemma 6.2. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables such that $X_n \leq D_n$ a.s. and $\mathbb{E}[X_n] = 2^{(\chi+o_n(1))n}$. For each $u \in \mathbb{N}$,

$$\mathbb{P}\left[X_n \geq 2^{(\chi-u)n}\right] \geq 2^{-o_n(n)}$$

at a rate depending only the law of the X_n 's, u , and γ .

Proof. Let $\zeta \in (0, u)$ and let

$$E_n := \left\{D_n \leq 2^{(\chi+\zeta)n}\right\}.$$

By Proposition 5.2,

$$\mathbb{P}[E_n^c] = o_n^\infty(2^n),$$

at a rate depending only on u and ζ . Since $X_n \leq D_n \leq 2^n$ and $\mathbb{E}[X_n] \geq 2^{(\chi+o_n(1))n}$, for large enough $n \in \mathbb{N}$ (depending on ζ),

$$\mathbb{E}[X_n \mathbb{1}_{E_n}] = \mathbb{E}[X_n] - \mathbb{E}[X_n \mathbb{1}_{E_n^c}] \geq 2^{(\chi+o_n(1))n}. \quad (6.2)$$

We have

$$\mathbb{E}[X_n \mathbb{1}_{E_n}] \leq 2^{(\chi-u)n} + \left(2^{(\chi+\zeta)n} - 2^{(\chi-u)n}\right) \mathbb{P}\left[X_n \geq 2^{(\chi-u)n}\right].$$

By combining this with (6.2), we find that for large enough $n \in \mathbb{N}$,

$$\mathbb{P}\left[X_n \geq 2^{(\chi-u)n}\right] \geq \frac{2^{(\chi+o_n(1))n} - 2^{(\chi-u)n}}{2^{(\chi+\zeta)n} - 2^{(\chi-u)n}} \geq 2^{-(\zeta-o_n(1))n}.$$

Since $\zeta \in (0, u)$ can be made arbitrarily small, we conclude. \square

Next we need to transfer the estimate of Proposition 5.2 to an estimate for the size of metric balls in $\mathcal{G}^{2^{-n}}|_{(0,1]}$.

Lemma 6.3. Let χ be as in Proposition 5.1 and let $\zeta \in (0, \chi/2)$. For $n \in \mathbb{N}$, let $E_n = E_n(\zeta)$ be the event that the following is true. For each $x \in (0, 1]_{2^{-n}\mathbb{Z}}$ and each $m \in [\zeta n, n]_{\mathbb{Z}}$, we have

$$\#\mathcal{B}_{2^m}(x; \mathcal{G}^{2^{-n}}|_{(0,1]}) \geq 2^{\frac{m}{\chi+\zeta}}.$$

There is a constant $c > 0$ depending only on γ such that for $n \in \mathbb{N}$,

$$\mathbb{P}[E_n^c] \preceq \exp(-c\zeta^2 n^2)$$

with the implicit constant depending only on ζ and γ .

Proof. For $r \in [\chi\zeta n - 1, n]_{\mathbb{Z}}$ and $x \in (2^{r-n}, 1]_{2^{-n}\mathbb{Z}}$, let

$$E_r^-(x) := \left\{\text{diam}\left(\mathcal{G}^{2^{-n}}|_{(x-2^{r-n}, x]}\right) \leq 2^{(\chi+\zeta)r}\right\}.$$

For $x \in (0, 1 - 2^{r-n}]_{2^{-n}\mathbb{Z}}$, also let

$$E_r^+(x) := \left\{\text{diam}\left(\mathcal{G}^{2^{-n}}|_{(x, x+2^{r-n}]}\right) \leq 2^{(\chi+\zeta)r}\right\}.$$

Let

$$\tilde{E}_n := \bigcap_{r=\lfloor \chi\zeta n \rfloor}^n \left(\bigcap_{x \in (2^{r-n}, 1]_{2^{-n}\mathbb{Z}}} E_r^-(x) \cap \bigcap_{x \in (0, 1-2^{r-n}]_{2^{-n}\mathbb{Z}}} E_r^+(x) \right).$$

By Proposition 5.2 and the union bound, we can find $c > 0$ depending only on γ such that

$$\mathbb{P}[\tilde{E}_n^c] \preceq \exp(-c\zeta^2 n^2)$$

with the implicit constant depending only on ζ and γ .

Now we will show that $\tilde{E}_n \subset E_n$. Suppose that \tilde{E}_n occurs and we are given $x \in (0, 1]_{2^{-n}\mathbb{Z}}$ and $m \in [\zeta n, n]_{\mathbb{Z}}$. Set $r := \lceil m/(\chi - \zeta) \rceil$. By symmetry we can assume without loss of generality that $x > 2^{r-n}$. Since \tilde{E}_n occurs, the event $E_r^-(x)$ occurs, so each element of $(x - 2^{r-n}, x]_{2^{-n}\mathbb{Z}}$ lies at graph distance at most $2^{(\chi+\zeta)r} \leq 2^m$ from x in $\mathcal{G}^{2^{-n}}|_{(0,1]}$. Therefore,

$$\#\mathcal{B}_{2^m}(x; \mathcal{G}^{2^{-n}}|_{(0,1]}) \geq 2^r \geq 2^{\frac{m}{\chi+\zeta}}. \quad \square$$

Proof of Proposition 6.1. It is clear that $X_n \leq D_n$ for each of the two possible choices of X_n in the statement of the lemma, so for each such choice we only need to prove that the limit in (6.1) is at least χ . We treat the two cases separately.

Case 1. Fix $u, \zeta \in (0, \chi/2)$ and let $E_n = E_n(\zeta)$ be as in Lemma 6.3. Suppose that $\{D_n \geq 2^{(\chi-u)n}\} \cap E_n$ occurs. Since $\{D_n \geq 2^{(\chi-u)n}\}$, for any given choice of $x_0^n \in (0, 1]_{2^{-n}\mathbb{Z}}$, there is a $y_0^n \in (0, 1]_{2^{-n}\mathbb{Z}}$ with

$$\text{dist}(x_0^n, y_0^n; \mathcal{G}^{2^{-n}}|_{(0,1]}) \geq 2^{(\chi-u)n-1}.$$

Let $m := \lfloor (\chi - u)n \rfloor - 2$. By the triangle inequality,

$$\text{dist}(x_0^n, y; \mathcal{G}^{2^{-n}}|_{(0,1]}) \geq 2^{(\chi-u)n-2}, \quad \forall y \in \mathcal{B}_{2^m}(y_0^n; \mathcal{G}^{2^{-n}}|_{(0,1]}).$$

Since E_n occurs, there are at least $2^{m/(\chi+\zeta)-1} \succeq 2^{n(\chi-u)/(\chi+\zeta)}$ elements of $\mathcal{B}_{2^m}(y_0^n; \mathcal{G}^{2^{-n}}|_{(0,1]})$. Since x_1^n is sampled uniformly from $(0, 1]_{2^{-n}\mathbb{Z}}$, we infer that

$$\mathbb{P}\left[\text{dist}(x_0^n, x_1^n; \mathcal{G}^{2^{-n}}|_{(0,1]}) \geq 2^{(\chi-u)n-2} \mid \mathcal{G}^{2^{-n}}|_{(0,1]}\right] \mathbb{1}_{\{D_n \geq 2^{(\chi-u)n}\} \cap E_n} \succeq 2^{-(\frac{\chi-u}{\chi+\zeta}-1)n} \mathbb{1}_{\{D_n \geq 2^{(\chi-u)n}\} \cap E_n} \quad (6.3)$$

with the implicit constant depending only on u, ζ , and γ . By Lemmas 6.2 and 6.3, for sufficiently large $n \in \mathbb{N}$ we have

$$\mathbb{P}\left[D_n \geq 2^{(\chi-u)n}, E_n\right] \geq 2^{-o_n(n)}.$$

Taking the expectation of both sides of (6.3) now yields

$$\mathbb{E}\left[\text{dist}(x_0^n, x_1^n; \mathcal{G}^{2^{-n}}|_{(0,1]})\right] \geq 2^{(\chi-u+\frac{\chi-u}{\chi+\zeta}-1-o_n(1))n}.$$

We obtain the lower bound in (6.1) for $X_n = \text{dist}(x_0^n, x_1^n; \mathcal{G}^{2^{-n}}|_{(0,1]})$ by sending $u \rightarrow 0$ and $\zeta \rightarrow 0$.

Case 2. By Lemmas 2.4 and 2.5, for each fixed $x, y \in (0, 1]_{2^{-n}\mathbb{Z}}$, we have

$$\mathbb{E}\left[\text{dist}(x, y; \mathcal{G}^{2^{-n}}|_{(0,1]})\right] \leq \mathbb{E}\left[\text{dist}(x, y; \mathcal{G}^{2^{-n}}|_{(x,y]})\right] \leq n\mathbb{E}\left[\text{dist}(2^{-n}, 1; \mathcal{G}^{2^{-n}}|_{(0,1]})\right].$$

Therefore for a deterministic choice of $x_0^n \in (0, 1]_{2^{-n}\mathbb{Z}}$ and a uniformly random choice of $x_1^n \in (0, 1]_{2^{-n}\mathbb{Z}}$ we have by case 1 that

$$\mathbb{E}\left[\text{dist}(2^{-n}, 1; \mathcal{G}^{2^{-n}}|_{(0,1]})\right] \geq \frac{1}{n}\mathbb{E}\left[\text{dist}(x_0^n, x_1^n; \mathcal{G}^{2^{-n}}|_{(0,1]})\right] \geq 2^{(\chi+o_n(1))n}.$$

This proves the lower bound in (6.1) for $X_n = \text{dist}(2^{-n}, 1; \mathcal{G}^{2^{-n}}|_{(0,1]})$. \square

6.2 Non-dyadic cell counts

In this subsection we will extend the results of Sections 5 and 6.1 to the case when the number of cells in the structure graph we are considering may not be a power of 2. By scale invariance it suffices to consider a general integer number of cells with unit mass.

Proposition 6.4. *There is a constant $c > 0$, depending only on γ , such that for each $u > 0$ and each $N \in \mathbb{N}$,*

$$\mathbb{P}[\text{diam}(\mathcal{G}^1|_{(0,N]}) > N^{\chi+u}] \preceq \exp(-cu^2(\log N)^2) \quad (6.4)$$

with the implicit constant depending only on u and γ . Furthermore, for each $u > 0$ and each $N \in \mathbb{N}$,

$$\mathbb{P}[\text{dist}(1, N; \mathcal{G}^1|_{(0,N]}) \geq N^{\chi-u}] \geq N^{-o_N(1)} \quad (6.5)$$

at a rate depending only on u and γ .

Proof. We first deduce the upper bound (6.4) from Proposition 5.2 in a similar manner to the proof of Lemma 2.5. Let $m := \lfloor \log_2 N \rfloor$. Choose $n_1, \dots, n_k \in [0, m]_{\mathbb{Z}}$ with $n_1 < \dots < n_k$ and $N = \sum_{j=1}^k 2^{n_j}$. We can write $(0, T]_{\mathbb{Z}} = \bigsqcup_{j=1}^k I_j$, where I_1, \dots, I_j are disjoint and each I_j is the intersection of \mathbb{Z} with some interval and satisfies $\#I_j = 2^{n_j}$. Then

$$\text{diam}(\mathcal{G}^1|_{(0,N]}) \leq \sum_{j=1}^k \text{diam}(\mathcal{G}^1|_{I_j}).$$

The random variables $\text{diam}(\mathcal{G}^1|_{I_j})$ are independent and by Lemma 2.4 (along with translation and scale invariance) each is stochastically dominated by a random variable with the same law as $\mathcal{G}^{2^{-m}}|_{(0,1]}$. We thus obtain the estimate (6.4) from Proposition 5.2 and the union bound.

To prove (6.5), fix $\zeta \in (0, 1)$ and choose $n \in \mathbb{N}$ such that $2^n \leq N \leq 2^{n+1}$. We will prove an upper bound for $\text{dist}(1, 2^{\lfloor (1+\zeta)n \rfloor}; \mathcal{G}^1|_{(0, 2^{\lfloor (1+\zeta)n \rfloor}]})$ in terms of $\text{dist}(1, N; \mathcal{G}^1|_{(0,N]})$ by decomposing $[1, 2^{\lfloor (1+\zeta)n \rfloor}]$ as a disjoint union of intervals of length N plus a small error interval over which the diameter of the structure graph is negligible.

Let $k := \lfloor N^{-1} 2^{\lfloor (1+\zeta)n \rfloor} \rfloor$ and note that $k \geq 2^{\zeta n-1}$. For $j \in [1, k]_{\mathbb{Z}}$, let

$$X_j := \text{dist}((j-1)N + 1, jN; \mathcal{G}^1|_{((j-1)N, jN]}).$$

Also let

$$Y := \text{dist}(kN, 2^{\lfloor (1+\zeta)n \rfloor}; \mathcal{G}^1|_{(kN, 2^{\lfloor (1+\zeta)n \rfloor}]})$$

Then the random variables X_1, \dots, X_k are iid, each has the same law as $\text{dist}(1, N; \mathcal{G}^1|_{(0,N]})$, and

$$\text{dist}(1, 2^{\lfloor (1+\zeta)n \rfloor}; \mathcal{G}^1|_{(0, 2^{\lfloor (1+\zeta)n \rfloor}]}) \leq \sum_{j=1}^k X_j + Y. \quad (6.6)$$

Let $v \in (0, u/2)$ be chosen so that $(1+\zeta)(\chi-v) > (1+\zeta/2)\chi$. By Proposition 6.1 and Lemma 6.2, for each $v > 0$ it holds with probability at least $2^{-o_n(n)}$ that

$$\text{dist}(1, 2^{\lfloor (1+\zeta)n \rfloor}; \mathcal{G}^1|_{(0, 2^{\lfloor (1+\zeta)n \rfloor}]}) \geq 2^{(1+\zeta)(\chi-v)n}. \quad (6.7)$$

Furthermore, by (6.4) it holds with probability at least $1 - o_n^\infty(2^n)$ that

$$Y \leq 2^{(1+\zeta/2)\chi n}. \quad (6.8)$$

By our choice of v , for large enough n the right side of (6.7) is at least twice the right side of (6.8). By (6.6), for large enough N , whenever both (6.7) and (6.8) occur, there must exist $j \in [1, k]_{\mathbb{Z}}$ such that $X_j \geq k^{-1} 2^{(1+\zeta)(\chi-v)n-1}$. By symmetry and the union bound, for each sufficiently large N ,

$$\mathbb{P}[X_1 \geq 2^{((1+\zeta)(\chi-v)-\zeta)n-1}] \geq 2^{-(\zeta+o_n(1))n}.$$

Since $v < u/2$, sending ζ to 0 yields (6.5). □

6.3 Proof of Theorems 1.12 and Theorem 1.15

We now conclude the proofs of Theorems 1.12 and 1.15.

Proof of Theorem 1.12. It is immediate from Proposition 6.4 and scale invariance that

$$\mathbb{E}[\text{diam}(\mathcal{G}^\epsilon|_{(0,1)})] = \epsilon^{-\chi+o_\epsilon(1)}.$$

Therefore (1.7) holds. The lower bound for χ follows from Proposition 3.1. By Lemma 2.8 we have $\chi \leq 1/2$ (since $\partial_\epsilon(0,1]$ contains a path from ϵ to 1 in $\mathcal{G}^\epsilon|_{(0,1]}$). \square

Proof of Theorem 1.15. The estimate (1.12) follows from (6.4) of Proposition 6.4. In the case when $s = 0$ and $t = 1$, the estimate (1.13) follows from (6.5). It remains to prove (1.13) for general $s, t \in [0,1]$ with $s < t$. To this end, fix such an s and t and let $x_s^\epsilon \approx s$ and $x_t^\epsilon \approx t$ be as in the theorem statement. We will prove (1.13) by showing that $\mathcal{G}^\epsilon|_{(0,1]}$ has “pinch points” at x_s^ϵ and x_t^ϵ on an event of probability decaying slower than an arbitrarily small power of ϵ , in which case $\text{dist}(x_s^\epsilon, x_t^\epsilon; \mathcal{G}^\epsilon|_{(0,1]})$ is close to $\text{dist}(x_s^\epsilon, x_t^\epsilon; \mathcal{G}^\epsilon|_{(x_s^\epsilon, x_t^\epsilon]})$.

Fix $u, \zeta > 0$ and for $\epsilon > 0$, let E_ϵ be the event that the following is true.

1. $\text{dist}(x_s^\epsilon, x_t^\epsilon; \mathcal{G}^\epsilon|_{[x_s^\epsilon, x_t^\epsilon]}) \geq 2\epsilon^{-\chi+\zeta}$.
2. Let y_s^ϵ be the closest element of $(0,1]_{\epsilon\mathbb{Z}}$ to $x_s^\epsilon + \epsilon^\zeta$ and let y_t^ϵ be the closest element of $(0,1]_{\epsilon\mathbb{Z}}$ to $x_t^\epsilon - \epsilon^\zeta$. Then $\text{diam}(\mathcal{G}^\epsilon|_{[x_s^\epsilon, y_s^\epsilon]})$ and $\text{diam}(\mathcal{G}^\epsilon|_{[y_t^\epsilon, x_t^\epsilon]})$ are each at most $\epsilon^{-(1-\zeta)(\chi-\zeta)}$.
3. In the notation of Definition 2.2, each of $\underline{\Delta}_{[x_s^\epsilon, y_s^\epsilon]}^L$, $\underline{\Delta}_{[x_s^\epsilon, y_s^\epsilon]}^R$, $\overline{\Delta}_{[y_t^\epsilon, x_t^\epsilon]}^L$, and $\overline{\Delta}_{[y_t^\epsilon, x_t^\epsilon]}^R$ is at least $\epsilon^{\zeta(1/2+\zeta)}$.

By scale invariance and the case when $s = 0$ and $t = 1$, the probability that condition 1 holds is at least $\epsilon^{o_\epsilon(1)}$. By (1.12), the probability that condition 2 fails to hold is of order $o_\epsilon^\infty(\epsilon)$. By standard estimates for Brownian motion, the probability that condition 3 fails to hold decays polynomially as $\epsilon \rightarrow 0$. Therefore, $\mathbb{P}[E_\epsilon] \geq \epsilon^{o_\epsilon(1)}$.

Let F_ϵ be the event that each of the following quantities is at most $\epsilon^{\zeta(1/2+\zeta)}$:

$$\overline{\Delta}_{[0, x_s^\epsilon]}^L, \quad \overline{\Delta}_{[0, x_s^\epsilon]}^R, \quad \underline{\Delta}_{[x_t^\epsilon, 1]}^L, \quad \text{and} \quad \underline{\Delta}_{[x_t^\epsilon, 1]}^R.$$

By Lemma 2.7 and the Markov property,

$$\mathbb{P}[F_\epsilon \cap E_\epsilon] \geq \epsilon^{\frac{8}{\gamma^2}\zeta(1/2+\zeta)+o_\epsilon(1)}. \quad (6.9)$$

Suppose now that $E_\epsilon \cap F_\epsilon$ occurs. By condition 3 in the definition of E_ϵ and the definition of F_ϵ , the only elements of $[x_s^\epsilon, x_t^\epsilon]_{\epsilon\mathbb{Z}}$ which are adjacent to an element of $(0, x_s^\epsilon)_{\epsilon\mathbb{Z}}$ (resp. $(x_t^\epsilon, 1]_{\epsilon\mathbb{Z}}$) are those in $[x_s^\epsilon, y_s^\epsilon]_{\epsilon\mathbb{Z}}$ (resp. $[y_t^\epsilon, x_t^\epsilon]_{\epsilon\mathbb{Z}}$). Furthermore, no element of $(0, x_s^\epsilon)_{\epsilon\mathbb{Z}}$ is adjacent to an element of $(x_t^\epsilon, 1]_{\epsilon\mathbb{Z}}$. By conditions 1 and 2 in the definition of E_ϵ , the distance from $(0, x_s^\epsilon)_{\epsilon\mathbb{Z}}$ to $(x_t^\epsilon, 1]_{\epsilon\mathbb{Z}}$ in $\mathcal{G}^\epsilon|_{(0,1]}$ is at least $2\epsilon^{-\chi+\zeta} - 2\epsilon^{(1-\zeta)(\chi-\zeta)}$, which is at least $\epsilon^{-\chi+\zeta} \geq \epsilon^{-\chi+u}$ for small enough ϵ . Since ζ is arbitrary, the estimate (1.13) follows from (6.9). \square

7 Distance to the lower boundary for $\gamma = \sqrt{2}$

In the remainder of this paper we will restrict attention to the case $\gamma = \sqrt{2}$, with the eventual aim of proving Theorem 1.16. In this section we will prove a lower bound for the distance from 1 to the lower boundary of $\mathcal{G}^\epsilon|_{(0,1]}$ (this improves on Proposition 6.4 in the case $\gamma = \sqrt{2}$, which only gives a lower bound for the distance from 1 to ϵ).

Proposition 7.1. *Suppose $\gamma = \sqrt{2}$. Let χ be as in Proposition 5.1 and let $u \in (0,1)$. For each $\epsilon > 0$ with $1/\epsilon \in \mathbb{N}$ we have*

$$\mathbb{P}[\text{dist}(1, \partial_\epsilon(0,1]; \mathcal{G}^\epsilon|_{(0,1]}) \geq \epsilon^{-\chi+u}] \geq \epsilon^{o_\epsilon(1)}. \quad (7.1)$$

with $\partial_\epsilon(0,1]$ as in Definition 2.1 and the rate of the $o_\epsilon(1)$ depending only on u .

Remark 7.2. The only step of the proof of Proposition 7.1 which does not work (with minor modifications) for general values of $\gamma \in (0, 2)$ is the invariance of the law of a two-dimensional correlated Brownian excursion under re-rooting (Lemma 7.6), which is true only in the case when the correlation is 0. All of the proofs in Section 8 are valid whenever $\gamma \in (0, 2)$ is such that $\chi > 1 - 2/\gamma^2$. Hence Lemma 7.6 is the sole reason why the proof of Theorem 1.16 does not work for general values of γ satisfying $\chi > 1 - 2/\gamma^2$.

For the proof of Proposition 7.1, we will need to consider several different conditioned Brownian motions and their associated structure graphs. For any such Brownian motion, the associated structure graph with cell size ϵ is defined in the same manner as the structure graph \mathcal{G}^ϵ in Section 1.3, with adjacency defined by one of the equivalent conditions (1.4) or (1.5) (but with the conditioned Brownian motion in question in place of Z).

Throughout this section, we let α be the variance of L and R , as in (2.2). Let $\dot{Z} = (\dot{L}, \dot{R}) : [0, 1] \rightarrow [0, \infty)^2$ be a pair of independent Brownian excursions started from 0 with variance α , i.e. an uncorrelated two-dimensional Brownian motion with variance α conditioned to stay in the first quadrant for one unit of time and satisfy $\dot{Z}(1) = 0$. For $\epsilon > 0$, let $\dot{\mathcal{G}}^\epsilon$ be the ϵ -structure graph associated with \dot{Z} .

We also let $\widehat{Z} = (\widehat{L}, \widehat{R})$ be a pair of independent Brownian meanders with variance α , i.e. an uncorrelated two-dimensional Brownian motion with variances α started from 0 and conditioned to stay in the first quadrant until time 1 (but not to end up at 0). For $\epsilon > 0$, let $\widehat{\mathcal{G}}^\epsilon$ be the ϵ -structure graph associated with \widehat{Z} .

We note that the coordinates of each of \dot{Z} and \widehat{Z} are independent, so many facts about \dot{Z} and \widehat{Z} can be readily derived from facts about one-dimensional Brownian excursions and meanders. In particular, one has Markov properties for \dot{Z} and \widehat{Z} .

Notation 7.3. For $z \in \mathbb{C}$, we write \mathbb{P}^z for the law of an (unconditioned) uncorrelated two-dimensional Brownian motion with variances α started from z .

7.1 Re-rooting

The key input in the proof of Proposition 7.1 is a certain symmetry of the structure graph $\dot{\mathcal{G}}^\epsilon$ which is not apparent from the definition, but is nevertheless quite natural if one is familiar with continuum random trees and/or the Brownian map (see [Le 14, Mie09] and the references therein). To describe this symmetry, we first need to introduce some notation.

Notation 7.4. For $t \in \mathbb{R}$, we write $\text{frac}(t) := t - \lfloor t \rfloor$ for the fractional part of t . For $s, t \in \mathbb{R}$, we write $s \oplus t := \text{frac}(s + t)$.

Notation 7.5. For $s, t \in \mathbb{R}$ and a function $f : \mathbb{R} \rightarrow \mathbb{R}$, we write $\mathfrak{m}^f(s, t) := \inf_{r \in [s \wedge t, s \vee t]} f(r)$.

For $\mathfrak{t} \in [0, 1]$ and $t \in [0, 1]$, let

$$\begin{aligned} \dot{L}_t^\mathfrak{t} &:= \dot{L}_t + \dot{L}_{t \oplus \mathfrak{t}} - 2\mathfrak{m}^{\dot{L}}(\mathfrak{t}, t \oplus \mathfrak{t}) \\ \dot{R}_t^\mathfrak{t} &:= \dot{R}_t + \dot{R}_{t \oplus \mathfrak{t}} - 2\mathfrak{m}^{\dot{R}}(\mathfrak{t}, t \oplus \mathfrak{t}) \\ \dot{Z}_t^\mathfrak{t} &:= (\dot{L}_t^\mathfrak{t}, \dot{R}_t^\mathfrak{t}). \end{aligned} \tag{7.2}$$

If we think of \dot{L} and \dot{R} as the contour functions of two rooted continuum random trees [Ald91a, Ald91b, Ald93], then $\dot{L}^\mathfrak{t}$ and $\dot{R}^\mathfrak{t}$ are the contour functions of the same trees, but with a different choice of marked point [LW06, Section 2.3]. For $N \in \mathbb{N}$ and $\mathfrak{x} \in (0, 1]_{N^{-1}\mathbb{Z}}$, let $\dot{\mathcal{G}}_\mathfrak{x}^{1/N}$ be the structure graph associated with $\dot{Z}^\mathfrak{t}$, with cells having area 1. We observe that the operation $\dot{Z} \mapsto \dot{Z}^\mathfrak{t}$ is invertible.

Lemma 7.6. For each $\mathfrak{t} \in [0, 1]$ we have $(\dot{Z}^\mathfrak{t})^{1-\mathfrak{t}} = \dot{Z}$.

Proof. By symmetry it suffices to show that $(\dot{L}^\mathfrak{t})^{1-\mathfrak{t}} = \dot{L}$. This can be accomplished by direct calculation, but we give a more conceptual proof. For $\mathfrak{t} \in [0, 1]$, let $(\mathcal{T}_\mathfrak{t}, w_\mathfrak{t})$ be the continuum tree whose contour function is $\dot{L}^\mathfrak{t}$ (see [Le 05, Section 2]). The rooted tree $(\mathcal{T}_\mathfrak{t}, w_\mathfrak{t})$ determines and is determined by $\dot{L}^\mathfrak{t}$. Let $f_\mathfrak{t} : [0, 1] \rightarrow \mathcal{T}_\mathfrak{t}$ be the quotient map induced by $\dot{L}^\mathfrak{t}$. Then we have $(\mathcal{T}_\mathfrak{t}, w_\mathfrak{t}) = (\mathcal{T}_0, f(\mathfrak{t}))$ and $f_\mathfrak{t}(\cdot) = f_0(\text{frac}(\cdot + \mathfrak{t}))$. Applying these facts with $\dot{L}^\mathfrak{t}$ in place of \dot{L} and $1 - \mathfrak{t}$ in place of \mathfrak{t} shows that the rooted CRT associated with $(\dot{L}^\mathfrak{t})^{1-\mathfrak{t}}$ is equal to $(\mathcal{T}_\mathfrak{t}, f_\mathfrak{t}(1 - \mathfrak{t})) = (\mathcal{T}_0, w_0)$. Therefore $(\dot{L}^\mathfrak{t})^{1-\mathfrak{t}} = \dot{L}$. \square

The symmetry which we are interested in is described by the following two lemmas, which tell us that the structure graph $\dot{\mathcal{G}}^{1/N}$ is invariant under re-rooting.

Lemma 7.7. *Let $N \in \mathbb{N}$ and $\mathbb{x}, x_1, x_2 \in (0, 1]_{N^{-1}\mathbb{Z}}$. Then x_1 and x_2 are connected by an edge in $\dot{\mathcal{G}}^{1/N}$ if and only if $x_1 \oplus (1 - \mathbb{x})$ and $x_2 \oplus (1 - \mathbb{x})$ are connected by an edge in $\dot{\mathcal{G}}_{\mathbb{x}}^{1/N}$.*

Proof. To lighten notation, we write

$$s' := s \oplus (1 - \mathbb{x}), \quad \forall s \in [0, 1].$$

By invertibility of the operation $\dot{Z} \mapsto \dot{Z}^{\mathbb{x}}$ (Lemma 7.6), it suffices to show that if x_1 and x_2 are connected by an edge in $\dot{\mathcal{G}}^{1/N}$, then x'_1 and x'_2 are connected by an edge in $\dot{\mathcal{G}}_{\mathbb{x}}^{1/N}$. We assume without loss of generality that $x_1 < x_2$. By assumption, there is an $s_1 \in [x_1 - 1/N, x_1]$ and an $s_2 \in [x_2 - 1/N, x_2]$ for which either

$$\inf_{s \in [s_1, s_2]} \dot{L}_s = \dot{L}_{s_1} = \dot{L}_{s_2} \quad \text{or} \quad \inf_{s \in [s_1, s_2]} \dot{R}_s = \dot{R}_{s_1} = \dot{R}_{s_2}.$$

Assume without loss of generality that the former condition holds. We have $s'_i \in [x'_i - 1/N, x'_i]$ for each $i \in \{1, 2\}$, so it suffices to show that

$$\inf_{s \in [s'_1 \wedge s'_2, s'_1 \vee s'_2]} \dot{L}_s^{\mathbb{x}} = \dot{L}_{s'_1}^{\mathbb{x}} = \dot{L}_{s'_2}^{\mathbb{x}}. \quad (7.3)$$

This follows easily from the definition (7.2). \square

The following lemma is the only step of the proof of Theorem 1.16 which works only for $\gamma = \sqrt{2}$.

Lemma 7.8. *For each $\mathfrak{t} \in [0, 1]$, we have $\dot{Z}^{\mathfrak{t}} \stackrel{d}{=} \dot{Z}$.*

Proof. Since the coordinates of \dot{Z} are independent, this follows from [MM06, Proposition 4.9] applied to each coordinate separately. \square

7.2 Distance conditioned on empty lower boundary

In this subsection we will prove the following proposition using a re-rooting argument based on the results of Section 7.1 (see in particular the proof of Lemma 7.13). We recall the definitions of the conditioned Brownian motions \dot{Z} and \widehat{Z} and their associated structure graphs from the beginning of this section, as well as the probability measures \mathbb{P}^z from Notation 7.3.

Proposition 7.9. *For each $u \in (0, 1)$ and each $\epsilon \in (0, 1)$,*

$$\mathbb{P} \left[\text{dist} \left(\epsilon, 1; \widehat{\mathcal{G}}^{\epsilon} \Big|_{(0,1]} \right) \geq \epsilon^{-\chi+u} \right] \geq \epsilon^{o_{\epsilon}(1)} \quad (7.4)$$

with the rate of the $o_{\epsilon}(1)$ depending only on u .

Proposition 7.9 can be viewed as a conditioned version of Proposition 7.1, since the lower boundary of $\widehat{\mathcal{G}}^{\epsilon}$ is equal to the single vertex ϵ . To prove Proposition 7.9, we will use local absolute continuity considerations to transfer some known estimates for the structure graph \mathcal{G}^{ϵ} associated with the unconditioned Brownian motion Z to estimates for $\widehat{\mathcal{G}}^{\epsilon}$ and then to estimates for $\dot{\mathcal{G}}^{\epsilon}$. This is accomplished in Lemmas 7.10, 7.11, and 7.12. In Lemma 7.13 we will use the re-rooting lemmas from Section 7.1 to obtain an estimate for the structure graph associated with the initial segment of \dot{Z} (which has empty lower boundary). Finally, we will transfer this estimate back to \widehat{Z} , with Lemma 7.14 as an intermediate step. We start with a general lemma which allows us to compare probabilities for conditioned and unconditioned Brownian motions.

Lemma 7.10. *Fix $C > 1$ and $q > 0$. Also let $T \in [C^{-1}, C]$ and $z \in [C^{-1/2}, C^{1/2}]^2$. There is a $\delta_0 > 0$, depending only on C and q such that the following is true. Let $\delta \in (0, \delta_0)$ and let E be an event which is measurable with respect to $\sigma(Z|_{[0, \delta]})$ and which satisfies $\mathbb{P}^z[E] \geq \delta^q$. Then*

$$\mathbb{P}^z[E \mid Z([0, T]) \subset (0, \infty)^2] \succeq \mathbb{P}^z[E]$$

with the implicit constant depending only on C .

Proof. By Bayes' rule,

$$\mathbb{P}^z[E \mid Z([0, T]) \subset (0, \infty)^2] = \frac{\mathbb{P}^z[Z([0, T]) \subset (0, \infty)^2 \mid E] \mathbb{P}^z[E]}{\mathbb{P}^z[Z([0, T]) \subset (0, \infty)^2]}. \quad (7.5)$$

By the Gaussian tail bound,

$$\mathbb{P}^z[|Z_\delta - z| > |z|/2] = o_\delta^\infty(\delta)$$

at a rate depending only on C so since $\mathbb{P}[E] \geq \delta^q$, we can find $\delta_0 > 0$ as in the statement of the lemma such that for $\delta \in (0, \delta_0)$ and any event E as in the statement of the lemma,

$$\mathbb{P}^z[|Z_\delta - z| \leq |z|/2, E] \geq \frac{1}{2} \mathbb{P}^z[E].$$

On the event $\{\sup_{s \in [0, \delta]} |Z_s - z| \leq |z|/2\}$, we have

$$\mathbb{P}^z[Z([0, T]) \subset (0, \infty)^2 \mid Z|_{[0, \delta]}] \geq 1.$$

Therefore,

$$\mathbb{P}^z[Z([0, T]) \subset (0, \infty)^2 \mid E] \geq 1.$$

We have $\mathbb{P}^z[Z([0, T]) \subset (0, \infty)^2] \asymp 1$. The statement of the lemma therefore follows from (7.5). \square

Our next lemma transfers the lower bound for structure graph distances from Theorem 1.15 to the structure graph associated with a segment of the conditioned Brownian motion \hat{Z} , and also includes a regularity condition.

Lemma 7.11. *Recall the definition of \hat{Z} and its structure graph $\hat{\mathcal{G}}^\epsilon$ from the beginning of this section. For $C > 1$, $u > 0$, $N \in \mathbb{N}$, $\epsilon > 0$, and $x \in \epsilon\mathbb{Z}$, let*

$$\hat{E}_N^\epsilon(x) = \hat{E}_N^\epsilon(x; C, u) := \left\{ \text{dist}\left(x - (N-1)\epsilon, x; \hat{\mathcal{G}}^\epsilon|_{(x-N\epsilon, x)}\right) \geq N^{\chi-u} \right\} \cap \left\{ \hat{Z}_x \in [C^{-1}, C]^2 \right\}. \quad (7.6)$$

For each $\lambda \in (0, 1/2)$ and each $v \in (0, 1/2)$, it holds that

$$\mathbb{P}\left[\hat{E}_N^\epsilon(x)\right] \geq N^{-o_N(1)}, \quad \forall \epsilon \in (0, 1), \quad \forall N \in [\epsilon^{-v}, \epsilon^{-(1-v)}]_{\mathbb{Z}}, \quad \forall x \in [\lambda, 1-\lambda]_{\epsilon\mathbb{Z}}$$

with the rate of the $o_N(1)$ depending only on C , u , λ , and v .

Proof. Suppose given $\epsilon > 0$, $N \in [1, (\lambda/2)\epsilon^{-(1-v)}]_{\mathbb{Z}}$, and $x \in [\lambda, 1-\lambda]_{\epsilon\mathbb{Z}}$. Let $\tilde{C} \in (1, C)$ and let

$$\begin{aligned} E_0 &:= \left\{ \hat{Z}_{x-N\epsilon} \in [\tilde{C}^{-1}, \tilde{C}]^2 \right\} \\ E_1 &:= \left\{ \text{dist}\left(x - (N-1)\epsilon, x; \hat{\mathcal{G}}^\epsilon|_{(x-N\epsilon, x)}\right) \geq N^{\chi-u} \right\} \\ E_2 &:= \left\{ \hat{Z}_x \in [C^{-1}, C]^2 \right\}. \end{aligned}$$

By the Markov property of \hat{Z} , the conditional law of $\hat{Z}|_{[x-N\epsilon, 1]}$ given $\hat{Z}|_{[0, x-N\epsilon]}$ is the same as the law $\mathbb{P}^{\hat{Z}_{x-N\epsilon}}$ conditioned on the event that the path stays in $(0, \infty)^2$ until time $1 - (x - N\epsilon)$. Furthermore, the event E_1 is determined by $\hat{Z}|_{[x-N\epsilon, 1]}$. By the lower bound in Theorem 1.15,

$$\mathbb{P}^{\hat{Z}_{x-N\epsilon}}\left[\text{dist}\left(x - (N-1)\epsilon, x; \hat{\mathcal{G}}^\epsilon|_{(x-N\epsilon, x)}\right) \geq N^{\chi-u}\right] \geq N^{-o_N(1)}.$$

so by Lemma 7.10 (applied with $T = 1 - (x - N\epsilon)$ and $\delta = N\epsilon$),

$$\mathbb{P}[E_1 \mid E_0] \geq N^{-o_N(1)}.$$

By the Gaussian tail bound (note that $N\epsilon \leq \epsilon^v$),

$$\mathbb{P}[E_2^c \mid E_0] = o_\epsilon^\infty(\epsilon) = o_N^\infty(N).$$

Therefore,

$$\mathbb{P}[E_1 \cap E_2 \mid E_0] \geq N^{-o_N(1)}.$$

Since we clearly have $\mathbb{P}[E_0] \geq 1$, we obtain the statement of the lemma. \square

Now we transfer to the case of the Brownian motion \dot{Z} and its associated structure graph $\dot{\mathcal{G}}^\epsilon$.

Lemma 7.12. *For $C > 1$, $u \in (0, 1)$, $N \in \mathbb{N}$, $\epsilon > 0$, and $x \in \epsilon\mathbb{Z}$, define the event $\dot{E}_N^\epsilon(x) = \dot{E}_N^\epsilon(x; C, u)$ in the same manner as the event $\hat{E}_N^\epsilon(x; C, u)$ of Lemma 7.11 but with \dot{Z} and its associated structure graph $\dot{\mathcal{G}}^\epsilon$ in place of \hat{Z} and $\hat{\mathcal{G}}^\epsilon$. For each $\lambda, v \in (0, 1/2)$, it holds that*

$$\mathbb{P}\left[\dot{E}_N^\epsilon(x)\right] \geq N^{-o_N(1)}, \quad \forall \epsilon \in (0, 1), \quad \forall N \in [\epsilon^{-v}, \epsilon^{-(1-v)}]_{\mathbb{Z}}, \quad \forall x \in [\lambda, 1 - \lambda]_{\epsilon\mathbb{Z}}$$

with the rate of the $o_N(1)$ depending only on C , u , λ , and v .

Proof. Let \hat{Z} be as in Lemma 7.11. Recalling that the coordinates of \hat{Z} are independent Brownian meanders, we find that for each $t \in (0, 1)$, the laws of $\dot{Z}|_{[0,t]}$ and $\hat{Z}|_{[0,t]}$ are mutually absolutely continuous. More precisely, the Radon-Nikodym derivative of the law of $\dot{Z}|_{[0,t]}$ with respect to the law of $\hat{Z}|_{[0,t]}$ is given by $g_t(\dot{Z}(t))$, where

$$g_t(z) := \frac{|z|^2 \exp\left(-\frac{|z|^2}{2(1-t)}\right) \sin(2 \arg(z))}{c_t \mathbb{P}^z[Z([0, 1-t]) \subset (0, \infty)^2]}, \quad (7.7)$$

where c_t is a normalizing constant depending only on t which is bounded above and below by universal positive constants for t in compact subsets of $(0, 1)$. See, e.g., [DIM77, Equations (1.1) and (1.2)].

On the event $\hat{E}_N^\epsilon(x)$ for $x \in [\lambda, 1 - \lambda]_{\epsilon\mathbb{Z}}$, it holds that $g_t(\hat{Z}(t))$ is bounded above and below by positive constants depending only on C and λ . Consequently,

$$\mathbb{P}\left[\dot{E}_N^\epsilon(x)\right] = \mathbb{E}\left[g_t(\hat{Z}(t)) \mathbb{1}_{\hat{E}_N^\epsilon(x)}\right] \asymp \mathbb{P}\left[\hat{E}_N^\epsilon(x)\right].$$

The statement of the lemma therefore follows from Lemma 7.11. \square

The following lemma contains the most important idea of this section, where we apply the re-rooting invariance properties studied in Section 7.1.

Lemma 7.13. *For each $u, v \in (0, 1/2)$ and each $\epsilon > 0$ with $1/\epsilon \in \mathbb{N}$,*

$$\mathbb{P}\left[\text{dist}\left(\epsilon, N\epsilon; \dot{\mathcal{G}}^\epsilon|_{(0, N\epsilon]}\right) \geq N^{\chi-u}\right] \geq N^{-o_N(1)}, \quad \forall N \in [\epsilon^{-v}, \epsilon^{-(1-v)}]_{\mathbb{Z}}$$

with the rate of the $o_N(1)$ depending only on u , and v .

Proof. Let $\epsilon \in (0, 1/4)$ with $1/\epsilon \in \mathbb{N}$ and let $N \in [\epsilon^{-v}, \epsilon^{-(1-v)}]_{\mathbb{Z}}$. Also fix $\mathfrak{x} \in [1/4, 3/4]_{\epsilon\mathbb{Z}}$. Let $\dot{Z}^\mathfrak{x}$ be defined as in (7.2) and let $\dot{\mathcal{G}}_\mathfrak{x}^\epsilon$ be its associated structure graph. By Lemma 7.8, we have $\dot{\mathcal{G}}_\mathfrak{x}^\epsilon \stackrel{d}{=} \dot{\mathcal{G}}^\epsilon$. Furthermore, Lemma 7.7 implies that the graphs

$$\dot{\mathcal{G}}_\mathfrak{x}^\epsilon|_{(0, N\epsilon]} \quad \text{and} \quad \dot{\mathcal{G}}^\epsilon|_{(\mathfrak{x}, \mathfrak{x} + N\epsilon]}$$

are isomorphic via the map which sends $x \in (0, N\epsilon]_{\epsilon\mathbb{Z}}$ to $x + \mathfrak{x}$. In particular,

$$\text{dist}\left(\epsilon, N\epsilon; \dot{\mathcal{G}}_\mathfrak{x}^\epsilon|_{(0, N\epsilon]}\right) = \text{dist}\left(\mathfrak{x} + \epsilon, \mathfrak{x} + N\epsilon; \dot{\mathcal{G}}^\epsilon|_{(\mathfrak{x}, \mathfrak{x} + N\epsilon]}\right). \quad (7.8)$$

Now fix $C > 1$ and let $\dot{E}_N^\epsilon(\mathfrak{x}) = \dot{E}_N^\epsilon(\mathfrak{x}; C, u)$ be as in Lemma 7.12. By (7.8) and Lemma 7.12,

$$\mathbb{P}\left[\text{dist}\left(\epsilon, N\epsilon; \dot{\mathcal{G}}^\epsilon|_{(0, N\epsilon]}\right) \geq N^{\chi-u}\right] \geq \mathbb{P}\left[\dot{E}_N^\epsilon(\mathfrak{x})\right] \geq N^{-o_N(N)}. \quad \square$$

We next prove a variant of Lemma 7.13 for the two-dimensional Brownian meander, rather than the two-dimensional Brownian excursion.

Lemma 7.14. *Recall the definition of \hat{Z} and $\hat{\mathcal{G}}^\epsilon$ from the beginning of this section. For each $u, v \in (0, 1/2)$ and each $\epsilon > 0$ with $1/\epsilon \in \mathbb{N}$,*

$$\mathbb{P}\left[\text{dist}\left(\epsilon, N\epsilon; \hat{\mathcal{G}}^\epsilon|_{(0, N\epsilon]}\right) \geq N^{\chi-u}\right] \geq N^{-o_N(1)}, \quad \forall N \in [\epsilon^{-v}, \epsilon^{-(1-v)}]_{\mathbb{Z}}$$

with the rate of the $o_N(1)$ depending only on u and v .

Proof. Define events

$$\begin{aligned}\dot{E}_0 &:= \left\{ \text{dist}\left(\epsilon, N\epsilon; \dot{\mathcal{G}}^\epsilon|_{(0, N\epsilon]}\right) \geq N^{\chi-u} \right\}, & \dot{E}_1 &:= \left\{ |\dot{Z}_{N\epsilon}| \leq 1 \right\} \\ \dot{E}_2 &:= \left\{ \dot{Z}_{1/2} \in [1/2, 1]^2 \right\}, & \dot{E} &:= \dot{E}_0 \cap \dot{E}_1 \cap \dot{E}_2.\end{aligned}$$

Define \hat{E} in the same manner as \dot{E} but with \hat{Z} in place of \dot{Z} . By the explicit formula for the density of the law of \dot{Z}_t with respect to Lebesgue measure (see [DIM77]), we have $\mathbb{P}[E_1^c] = o_N^\infty(N)$. By Lemma 7.13 we have $\mathbb{P}[\dot{E}_0] \geq N^{-o_N(1)}$, so

$$\mathbb{P}[\dot{E}_0 \cap \dot{E}_1] \geq N^{-o_N(1)}.$$

By the Markov property of \dot{Z} we have $\mathbb{P}[\dot{E}_2 | \dot{E}_0 \cap \dot{E}_1] \geq 1$ with universal implicit constant. Hence $\mathbb{P}[\dot{E}] \geq N^{-o_N(1)}$. On the event \dot{E} , we have $g_{1/2}(\dot{Z}_{1/2}) \asymp 1$ with universal implicit constant, where $g_{1/2}$ is the Radon-Nikodym derivative as in (7.7). Therefore, Lemma 7.13 implies that

$$\mathbb{P}[\hat{E}] = \mathbb{E}[g_{1/2}(\dot{Z}_{1/2}) \mathbb{1}_{\dot{E}}] \geq N^{-o_N(1)}. \quad \square$$

Proof of Proposition 7.9. Let $v > 0$ be a small parameter which we will eventually send to 0. Let $N := \lfloor 1/\epsilon \rfloor$ and let $n \in \mathbb{N}$ be chosen so that $2^{(1-\zeta)n-1} \leq N \leq 2^{(1-v)n}$. Let \tilde{Z} be an uncorrelated two-dimensional Brownian motion with variance α conditioned to stay in the first quadrant until time $N2^{-n}$ (so that \tilde{Z} evolves as a standard linear Brownian motion after time $N2^{-n}$). Let $\tilde{\mathcal{G}}^{2^{-n}}$ be the structure graph associated with \tilde{Z} , with cells of size 2^{-n} . By scale invariance, it suffices to show that

$$\mathbb{P}[E] \geq N^{-o_N(1)} \quad (7.9)$$

where

$$E := \left\{ \text{dist}\left(2^{-n}, N2^{-n}; \tilde{\mathcal{G}}^{2^{-n}}|_{(0, N2^{-n}]}\right) \geq N^{\chi-u} \right\}.$$

Let F be the event that $\tilde{Z}([N2^{-n}, 1]) \subset (0, \infty)^2$. Then the law of \hat{Z} is the same as the conditional law of \tilde{Z} given F . By Lemma 7.14 we have $\mathbb{P}[E | F] \geq N^{-o_N(1)}$. Since $N2^{-n} \geq 2^{-vn-1} \geq N^{-v}$ we have $\mathbb{P}[F] \geq N^{-o_v(1)}$ with universal implicit constant. Combining these assertions and letting $v \rightarrow 0$ yields (7.9). \square

7.3 Proof of Proposition 7.1

In this section we will deduce Proposition 7.1 from Proposition 7.9. Roughly, the idea of the proof is as follows. If we condition on $\hat{Z}|_{[0, \epsilon\zeta]}$ for a small but fixed $\zeta > 0$, then the conditional law of $(\hat{Z} - \hat{Z}_{\epsilon\zeta})|_{[\epsilon\zeta, 1]}$ is the same as the conditional law of $Z|_{[0, 1-\epsilon\zeta]}$ given that its two coordinates stay above $-\hat{L}_{\epsilon\zeta}$ and $-\hat{R}_{\epsilon\zeta}$, respectively. With high probability, $\hat{L}_{\epsilon\zeta}$ and $\hat{R}_{\epsilon\zeta}$ are at least $\epsilon^{\zeta(1-\zeta)/2}$, so the conditional law of $(\hat{Z} - \hat{Z}_{\epsilon\zeta})|_{[\epsilon\zeta, 1]}$ given $\hat{Z}|_{[0, \epsilon\zeta]}$ is not too much different from the unconditional law of $Z|_{[0, 1-\epsilon\zeta]}$. Furthermore, it is likely that every point on the lower boundary of the structure graph corresponding to $(\hat{Z} - \hat{Z}_{\epsilon\zeta})|_{[\epsilon\zeta, 1]}$ is close to ϵ . Hence the triangle inequality gives a lower bound for the distance from 1 to this lower boundary in terms of the distance in $\hat{\mathcal{G}}^\epsilon$ from 1 to ϵ .

For the proof we will need two elementary Brownian motion lemmas.

Lemma 7.15. *Let B be a standard linear Brownian motion, let $T > 0$, and let \underline{t} be the time at which B attains its infimum on the interval $[0, T]$. For each $\delta > 0$ and each $u \in (0, 1)$,*

$$\mathbb{P}\left[\underline{t} > \delta^{2-u} \mid \inf_{s \in [0, T]} B_s \geq -\delta\right] \leq \delta^{u/2+o_\delta(1)},$$

at a rate depending only on u , but uniform for u in compact subsets of $(0, 1)$.

Proof. Fix $\zeta > 0$. Let τ be the smallest $t > 0$ for which $B_t \geq \delta^{1-u/2+\zeta}$. It is easy to see from the Markov property that

$$\mathbb{P}\left[\tau > \delta^{2+u}, \inf_{s \in [0, \delta^{2+u}]} B_s \geq -\delta\right] = o_\delta^\infty(\delta).$$

Since $\mathbb{P}[\inf_{s \in [0, T]} B_s \geq -\delta] \succeq \delta$ and ζ is arbitrary, it suffices to show that

$$\mathbb{P}\left[\underline{t} > \delta^{2+u}, \tau \leq \delta^{2+u} \mid \inf_{s \in [0, T]} B_s \geq -\delta\right] \preceq \delta^{u/2-\zeta}. \quad (7.10)$$

Let σ be the smallest $t \geq \tau$ for which $B_s = 0$. Since $B_{\underline{t}} < 0$, on the event $\{\underline{t} > \delta^{2+u}\}$ we have $\sigma \leq 1$. By standard stochastic calculus (see, e.g., [Law05, Example 1.17]), the conditional law of $(B + \delta)|_{[\tau, 1]}$ given $B|_{[0, \tau]}$ and $\{\inf_{s \in [0, T]} B_s \geq -\delta\}$ is that of a 3-dimensional Bessel process started from $\delta + \delta^{1+u/2-\zeta}$. Let \widehat{B} be such a process. By Itô's formula, \widehat{B}^{-1} is a martingale, so for $C > 0$,

$$\mathbb{P}\left[\widehat{B} \text{ hits } \delta \text{ before } C\right] = \frac{(\delta + \delta^{1-u/2+\zeta})^{-1} - C^{-1}}{\delta^{-1} - C^{-1}}.$$

Sending $C \rightarrow \infty$, we find

$$\mathbb{P}\left[\sigma \leq 1 \mid \tau \leq \delta^{2+u}, \inf_{s \in [0, T]} B_s \geq -\delta\right] \preceq \delta^{u/2-\zeta}.$$

Therefore,

$$\mathbb{P}\left[\underline{t} > \delta^{2+u} \mid \tau \leq \delta^{2+u}, \inf_{s \in [0, T]} B_s \geq -\delta\right] \preceq \delta^{u/2-\zeta}$$

which implies (7.10) upon sending $\zeta \rightarrow 0$. \square

Lemma 7.16. For $\delta \in (0, 1)$, let \widehat{t}_δ^L (resp. \widehat{t}_δ^R) be the time at which \widehat{L} (resp. \widehat{R}) attains its minimum value on the interval $[\delta, 1]$. For each $\beta \in (0, 1)$, there exists $c > 0$ such that

$$\mathbb{P}[\widehat{t}_\delta^L > \delta^\beta \text{ or } \widehat{t}_\delta^R > \delta^\beta] \leq \delta^{c+o_\delta(1)}.$$

Proof. Since \widehat{L} and \widehat{R} are independent, it suffices to prove that there exists $c > 0$ such that

$$\mathbb{P}[\widehat{t}_\delta^L > \delta^\beta] \leq \delta^{c+o_\delta(1)}. \quad (7.11)$$

For $\zeta > 0$, the probability that \widehat{L}_δ does not belong to $[\delta^{1/2-\zeta}, \delta^{1/2+\zeta}]$ decays polynomially in δ . The conditional law of $(\widehat{L} - \widehat{L}_\delta)|_{[\delta, 1]}$ given \widehat{L}_δ is that of a Brownian motion conditioned to stay above $-\widehat{L}_\delta$. Therefore, the statement of the lemma follows from Lemma 7.15 (applied with \widehat{L}_δ in place of δ) upon making an appropriate choice of ζ (depending on β). \square

Proof of Proposition 7.1. Let $\epsilon \in (0, 1)$ and assume without loss of generality that $1/\epsilon \in \mathbb{Z}$. Also fix $\zeta, v \in (0, 1)$ to be chosen later, depending on u . Let $y_\epsilon := \epsilon \lceil \epsilon^{\zeta-1} \rceil$, so that $y_\epsilon \in (0, 1]_{\epsilon\mathbb{Z}}$. Let $\widetilde{Z}^\epsilon = (\widetilde{L}^\epsilon, \widetilde{R}^\epsilon)$ be an uncorrelated two-dimensional Brownian motion with variance α conditioned to stay in the first quadrant until time $1 + y_\epsilon$ and let $\widetilde{\mathcal{G}}^\epsilon$ be its associated structure graph with cells of size ϵ . Note that \widetilde{Z}^ϵ is defined in a similar manner to \widehat{Z} but conditioned to stay in the first quadrant for slightly more than one unit of time.

Let \widetilde{t}_ϵ^L and \widetilde{t}_ϵ^R be the times at which \widetilde{L}^ϵ and \widetilde{R}^ϵ , respectively, attain their minimum values on $[y_\epsilon, 1 + y_\epsilon]$. With $\underline{\partial}_\epsilon(y_\epsilon, 1 + y_\epsilon)$ the lower boundary of $\widetilde{\mathcal{G}}^\epsilon|_{[0, 1]}$ (Definition 2.1),

$$\underline{\partial}_\epsilon(y_\epsilon, 1 + y_\epsilon) \subset (y_\epsilon, \widetilde{t}_\epsilon^L \vee \widetilde{t}_\epsilon^R]_{\epsilon\mathbb{Z}}. \quad (7.12)$$

Fix $\beta \in (0, 1)$ and let

$$E^\epsilon := \left\{ \text{dist}\left(\epsilon, 1 + y_\epsilon; \widetilde{\mathcal{G}}^\epsilon|_{(0, 1+y_\epsilon)}\right) \geq \epsilon^{-\chi+v} \right\} \cap \left\{ \widetilde{t}_\epsilon^L \vee \widetilde{t}_\epsilon^R \leq \epsilon^{\beta\zeta} \right\} \cap \left\{ \widetilde{Z}_{y_\epsilon}^\epsilon \in \left[\epsilon^{\zeta(1+\zeta)/2}, \epsilon^{\zeta(1-\zeta)/2} \right]^2 \right\}.$$

By Lemma 7.9 and scale invariance, the probability of the first event in the definition of E^ϵ is at least $\epsilon^{o_\epsilon(1)}$. By Lemma 7.16 and [Shi85, Equation (3.2)], the probability that each of the other two events in the definition of E^ϵ fails to occur decays polynomially in ϵ . Therefore,

$$\mathbb{P}[E^\epsilon] \geq \epsilon^{o_\epsilon(1)}.$$

By applying the upper bound of Proposition 6.4 to each of the graphs $\tilde{\mathcal{G}}^\epsilon|_{(2^{-k}y_\epsilon, 2^{-k+1}y_\epsilon]}$ for $k \in \mathbb{N}$ such that $\epsilon \leq 2^{-k}y_\epsilon \leq \epsilon^{\beta\zeta}$ (in a similar manner to the argument at the end of the proof of Lemma 4.3), we find that except on an event of probability $o_\epsilon^\infty(\epsilon)$ we have

$$\text{dist}(\epsilon, x; \tilde{\mathcal{G}}^\epsilon|_{(0, y_\epsilon]}) \leq \epsilon^{-(1-\beta\zeta)(\chi+v)}, \quad \forall x \in (0, \epsilon^{\beta\zeta}]_{\mathbb{Z}}. \quad (7.13)$$

By (7.12), if E^ϵ occurs and (7.13) holds then

$$\text{dist}(\epsilon, x; \tilde{\mathcal{G}}^\epsilon|_{(0, 1+y_\epsilon]}) \leq \epsilon^{-(1-\beta\zeta)(\chi+v)}, \quad \forall x \in \tilde{\partial}_\epsilon(y_\epsilon, 1+y_\epsilon]. \quad (7.14)$$

Henceforth assume that v is chosen sufficiently small (depending on ζ) that $(1-\beta\zeta)(\chi+v) < \chi - v$.

By (7.14) and the triangle inequality, if ϵ is chosen sufficiently small (depending on ζ and v), then whenever E^ϵ occurs and (7.13) holds we have

$$\text{dist}(1+y_\epsilon, \tilde{\partial}_\epsilon(y_\epsilon, 1+y_\epsilon]; \tilde{\mathcal{G}}^\epsilon|_{(y_\epsilon, 1+y_\epsilon]}) \geq \frac{1}{2}\epsilon^{-\chi+v} \quad \text{and} \quad \tilde{Z}_{y_\epsilon}^\epsilon \in [\epsilon^{\zeta(1+\zeta)/2}, \epsilon^{\zeta(1-\zeta)/2}]^2. \quad (7.15)$$

Let \tilde{E}^ϵ be the event that (7.15) holds. Then $\mathbb{P}[\tilde{E}^\epsilon] \geq \epsilon^{o_\epsilon(1)} - o_\epsilon^\infty(\epsilon) \geq \epsilon^{o_\epsilon(1)}$. For each $\delta > 0$, we have

$$\epsilon^{o_\epsilon(1)} \leq \mathbb{P}[\tilde{E}^\epsilon] = \mathbb{E}[\mathbb{P}[\tilde{E}^\epsilon | \tilde{Z}^\epsilon|_{[0, y_\epsilon]}]] \leq \epsilon^\delta + \mathbb{P}[\mathbb{P}[\tilde{E}^\epsilon | \tilde{Z}^\epsilon|_{[0, y_\epsilon]}] \geq \epsilon^\delta].$$

By re-arranging, we find that for each $\delta > 0$,

$$\mathbb{P}[\mathbb{P}[\tilde{E}^\epsilon | \tilde{Z}^\epsilon|_{[0, y_\epsilon]}] \geq \epsilon^\delta] \geq \epsilon^{o_\epsilon(1)}. \quad (7.16)$$

The conditional law of $(\tilde{Z}^\epsilon - \tilde{Z}_{y_\epsilon}^\epsilon)|_{[y_\epsilon, 1+y_\epsilon]}$ given $\tilde{Z}^\epsilon|_{[0, y_\epsilon]}$ is the same as the law of $Z|_{[0, 1]}$ conditioned on the event that

$$\inf_{s \in [0, 1]} L_s \geq -\tilde{L}_{y_\epsilon}^\epsilon \quad \text{and} \quad \inf_{s \in [0, 1]} R_s \geq -\tilde{R}_{y_\epsilon}^\epsilon.$$

By (7.16) and the definition (7.15) of \tilde{E}^ϵ , for each $\delta > 0$ there exists $a_L, a_R \in [\epsilon^{\zeta(1+\zeta)/2}, \epsilon^{\zeta(1-\zeta)/2}]$ such that

$$\mathbb{P}\left[\text{dist}(1, \underline{\partial}_\epsilon(0, 1]; \mathcal{G}^\epsilon|_{(0, 1]}) \geq \epsilon^{-\chi+u} \mid \inf_{s \in [0, 1]} L_s \geq -a_L \text{ and } \inf_{s \in [0, 1]} R_s \geq -a_R\right] \geq \epsilon^\delta.$$

By Lemma 2.7 we have

$$\mathbb{P}\left[\inf_{s \in [0, 1]} L_s \geq -a_L \text{ and } \inf_{s \in [0, 1]} R_s \geq -a_R\right] \geq \epsilon^{\zeta(1+\zeta)}.$$

Therefore,

$$\mathbb{P}[\text{dist}(1, \underline{\partial}_\epsilon(0, 1]; \mathcal{G}^\epsilon|_{(0, 1]}) \geq \epsilon^{-\chi+u}] \geq \epsilon^{\delta+\zeta(1+\zeta)}.$$

Sending $\delta \rightarrow 0$ and $\zeta \rightarrow 0$ concludes the proof of the proposition. \square

8 Distance to the union of three boundary arcs for $\gamma = \sqrt{2}$

In this section, we will build on Proposition 7.1 to prove the following proposition, which immediately implies Theorem 1.16.

Proposition 8.1. *Suppose $\gamma = \sqrt{2}$ and fix $u > 0$. There is a constant $c > 0$, depending only on χ , such that for each $\epsilon \in (0, 1)$ we have (in the notation of Definition 2.1)*

$$\mathbb{P} \left[\max_{x \in (0, 1]_{\epsilon\mathbb{Z}}} \text{dist} \left(x, \underline{\partial}_\epsilon(0, 1] \cup \bar{\partial}_\epsilon(0, 1]; \mathcal{G}^\epsilon|_{(0, 1]} \right) \geq \epsilon^{-\chi+u} \right] \geq 1 - \epsilon^{cu+o_\epsilon(1)}.$$

Note that this is our first result which gives a lower bound on distance which holds with probability tending to 1 as $\epsilon \rightarrow 0$, rather than just with probability $\epsilon^{o_\epsilon(1)}$.

Remark 8.2. We have to require $\gamma = \sqrt{2}$ in Proposition 8.1 only because Proposition 7.1 requires $\gamma = \sqrt{2}$ (which is the case only because Lemma 7.8 requires $\gamma = \sqrt{2}$). If Proposition 7.1 were proven for all values of $\gamma \in (0, 2)$, our proof of Proposition 8.1 would work whenever $\gamma \in (0, 2)$ is such that $\chi > 1 - 2/\gamma^2$.

8.1 Distances in an excursion away from the running infimum

In this subsection we will prove the following proposition, which is the key input in the proof of Proposition 8.1.

Proposition 8.3. *Suppose $\gamma = \sqrt{2}$. Let σ^L be the smallest $t \geq 1$ for which the following is true. There exists $\tilde{t} \in [0, t - 1]$ such that*

$$L_t = L_{\tilde{t}} = \inf_{s \in [0, t]} L_s.$$

Let $\tilde{\sigma}^L$ be the largest corresponding value of \tilde{t} . For each $u > 0$ and each $\epsilon \in (0, 1)$, we have (in the notation of Definition 2.1)

$$\mathbb{P} \left[\max_{x \in [\tilde{\sigma}^L, \sigma^L]_{\epsilon\mathbb{Z}}} \text{dist} \left(x, \underline{\partial}_\epsilon[\tilde{\sigma}^L, \sigma^L]; \mathcal{G}^\epsilon|_{[\tilde{\sigma}^L, \sigma^L]} \right) \geq \epsilon^{-\chi+u}, \sigma^L \leq \epsilon^{-u} \right] \geq \epsilon^{o_\epsilon(1)}. \quad (8.1)$$

We note that in the setting of Proposition 8.3, it holds that $(L - L_{\tilde{\sigma}^L})|_{[\tilde{\sigma}^L, \sigma^L]}$ is the first excursion of L away from its running infimum which has time length at least 1. Since $L_{\sigma^L} = L_{\tilde{\sigma}^L} = \inf_{s \in [\tilde{\sigma}^L, \sigma^L]} L_s$, both $\underline{\partial}^L[\tilde{\sigma}^L, \sigma^L]$ and $\bar{\partial}^L[\tilde{\sigma}^L, \sigma^L]$ consist of a single vertex. Hence Proposition 8.3 can be viewed as an analogue of Proposition 8.1 for the structure graph $\mathcal{G}^\epsilon|_{[\tilde{\sigma}^L, \sigma^L]}$.

Proposition 8.3 is the only statement in this subsection which will be used in the proof of Proposition 8.1, so the reader can safely skip to Section 8.2 before reading the proof of Proposition 8.3.

The main step of the proof of Proposition 8.3, which is carried out in the proof of Lemma 8.6, is to find a path from ϵ to $\bar{\partial}_\epsilon(0, 1]$ in $\mathcal{G}^\epsilon|_{(0, 1]}$ which is a concatenation of paths in subgraphs of $\mathcal{G}^\epsilon|_{(0, 1]}$ which have the same law as $\mathcal{G}^\epsilon|_{[\tilde{\sigma}^L, \sigma^L]}$. This will lead to an upper bound for $\text{dist}(\epsilon, \bar{\partial}_\epsilon(0, 1]; \mathcal{G}^\epsilon|_{(0, 1]})$ in terms of the quantity considered in (8.1). Since we have a lower bound for $\text{dist}(\epsilon, \bar{\partial}_\epsilon(0, 1]; \mathcal{G}^\epsilon|_{(0, 1]})$ (Proposition 7.1), this will prove Proposition 8.3; c.f. the proof of the lower bound in Proposition 6.4 for another argument using a similar idea.

Our proof of Proposition 8.3 uses Proposition 7.1, so is only valid for $\gamma = \sqrt{2}$. However, in order to illustrate the source of the condition $\chi > 1 - 2/\gamma^2$ of Remark 8.2, we will give most of the argument assuming a general value of $\gamma \in (0, 2)$ which satisfies $\chi > 1 - 2/\gamma^2$.

We now introduce the notation which we will use for the proof of Proposition 7.1. For $r \geq 0$, let

$$T_r^L := \inf\{t \geq 0 : L_t < -r\} \quad \text{and} \quad \tilde{T}_r^L := \sup\{t < T_r^L : L_t = -r\}$$

Let $\rho_0^L := 0$ and for $k \in \mathbb{N}$, inductively let

$$\rho_k^L := \inf\left\{r > \rho_{k-1}^L : T_r^L - \tilde{T}_r^L \geq 1\right\}.$$

For $k \in \mathbb{N}_0$ let

$$\sigma_k^L := T_{\rho_k^L}^L \quad \text{and} \quad \tilde{\sigma}_k^L := \tilde{T}_{\rho_k^L}^L \quad (8.2)$$

be the endpoints of the k th interval of L away from its running infimum with time length at least 1. Define σ_k^R and $\tilde{\sigma}_k^R$ similarly but with R in place of L . The following lemma is an immediate consequence of the above definitions and the strong Markov property.

Lemma 8.4. *The excursions $(Z - Z_{\sigma_{k-1}^L})|_{[\sigma_{k-1}^L, \sigma_k^L]}$ are iid, and each has the same law as $Z|_{[0, \sigma^L]}$, with σ^L as in Lemma 8.3.*

We also have the following elementary Brownian motion estimate, which as we will see in the proof of Lemma 8.6 is the source of the condition $\chi > 1 - 2/\gamma^2$ mentioned in Remark 8.2 (see in particular (8.7)).

Lemma 8.5. *For $t \geq 1$, let F_t be the event that $[t-1, t]$ is not contained in the interval $[\tilde{\sigma}_k^q, \sigma_k^q]$ for any $q \in \{L, R\}$ or $k \in \mathbb{N}$. Then*

$$\mathbb{P}[F_t] \preceq t^{-2/\gamma^2},$$

with the implicit constant depending only on γ .

Proof. Assume without loss of generality that $t \geq 2$. Let \mathfrak{t}_t^L (resp. \mathfrak{t}_t^R) be the largest $s < t$ for which L (resp. R) attains a running infimum relative to time 0. By definition, if F_t occurs then we have $\mathfrak{t}_t^L \in [t-1, t]$ and $\mathfrak{t}_t^R \in [t-1, t]$. Let

$$X_t^L := \inf_{s \in [t-1, t]} (L_s - L_{t-1}) \quad \text{and} \quad X_t^R := \inf_{s \in [t-1, t]} (R_s - R_{t-1})$$

Then if F_t occurs,

$$\inf_{s \in [0, t-1]} (L_s - L_{t-1}) \geq X_t^L \quad \text{and} \quad \inf_{s \in [0, t-1]} (R_s - R_{t-1}) \geq X_t^R.$$

By Lemma 2.7, applied to the Brownian motion $s \mapsto Z_{t-1-s} - Z_{t-1}$, we find that the conditional probability given $(Z - Z_{t-1})|_{[t-1, t]}$ that this is the case is

$$\preceq (X_t^L \wedge X_t^R)(X_t^L \vee X_t^R)^{4/\gamma^2 - 1} t^{-2/\gamma^2}.$$

Averaging over all possible values of X_t^L and X_t^R now yields the statement of the lemma. \square

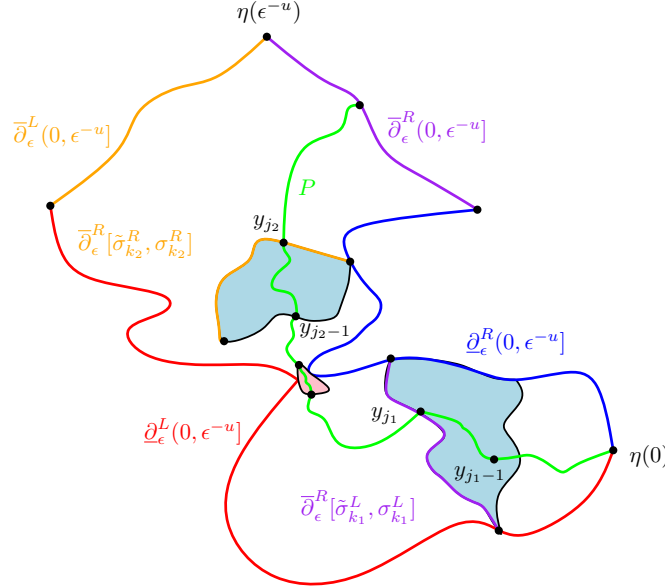


Figure 12: An illustration of the proof of Proposition 8.3. Shown is the curve segment $\eta([0, \epsilon^{-u}])$ with its four distinguished boundary arcs. We construct a path P (shown in green) from $\eta(0)$ to $\bar{\partial}_\epsilon(0, \epsilon^{-u})$ by concatenating paths from points of $\mathcal{G}^\epsilon|_{[\tilde{\sigma}_k^q, \sigma_k^q]}$ which terminate at $\bar{\partial}_\epsilon[\tilde{\sigma}_k^q, \sigma_k^q]$ for $k \in \mathbb{N}$ and $q \in \{L, R\}$. Two such paths are shown in the figure, which lie in the two blue regions $\eta([\tilde{\sigma}_{k_1}^L, \sigma_{k_1}^L])$ and $\eta([\tilde{\sigma}_{k_2}^R, \sigma_{k_2}^R])$. Whenever we are in a region which is not contained in any of the regions $\eta([\tilde{\sigma}_k^q, \sigma_k^q])$, we just follow cells in the order they are traced by η . Such regions correspond to “approximate local cut points” of η . One such region is shown in pink in the figure.

The following lemma is the main input in the proof of Proposition 8.3.

Lemma 8.6. *For each $v > 0$ and each $\epsilon \in (0, 1)$, we have (in the notation of Proposition 8.3)*

$$\mathbb{E} \left[\max_{x \in [\bar{\sigma}^L, \sigma^L]_{\epsilon\mathbb{Z}}} \text{dist}(x, \bar{\partial}_\epsilon[\bar{\sigma}^L, \sigma^L]; \mathcal{G}^\epsilon|_{[\bar{\sigma}^L, \sigma^L]}) \mathbb{1}_{(\sigma^L \leq \epsilon^{-v})} \right] \geq \epsilon^{-\chi + o_\epsilon(1)}. \quad (8.3)$$

Proof. Fix $v > u > \zeta > 0$ and $\epsilon > 0$ with $1/\epsilon \in \mathbb{N}$. To prove the lemma, we will construct a path from ϵ to $\bar{\partial}_\epsilon(0, \epsilon^{-u}]$ in $\mathcal{G}^\epsilon|_{(0, \epsilon^{-u}]}$ by iteratively concatenating paths in $\mathcal{G}^\epsilon|_{[\bar{\sigma}_k^q, \sigma_k^q]}$ which terminate at $\bar{\partial}_\epsilon[\bar{\sigma}_k^q, \sigma_k^q]$ for $k \in \mathbb{N}$ and $q \in \{L, R\}$. There are a small number of $y \in (0, \epsilon^{-u}]_{\epsilon\mathbb{Z}}$ which are not contained in any of these excursions. When our path runs into such an y we will instead travel to $y + 1$ along a path in $\mathcal{G}^\epsilon|_{[y, y+1]}$ of minimal length. The total length of these “bad” path segments will be bounded using Lemma 8.5 and the upper bound in Theorem 1.15. See Figure 12 for an illustration.

For $q \in \{L, R\}$, define $\bar{\sigma}_k^q$ and σ_k^q as above and for $t \geq 1$ define F_t as in Lemma 8.5. Let

$$K^q := \inf \{k \in \mathbb{N} : \sigma_k^q \geq \epsilon^{-u}\}$$

and

$$X_k^q := \max_{x \in [\bar{\sigma}_k^q, \sigma_k^q]_{\epsilon\mathbb{Z}}} \text{dist} \left(x, \bar{\partial}_\epsilon[\bar{\sigma}_k^q, \sigma_k^q]; \mathcal{G}^\epsilon|_{[\bar{\sigma}_k^q, \sigma_k^q]} \right) \mathbb{1}_{(\sigma_k^q - \sigma_{k-1}^q \leq \epsilon^{-v})} \quad (8.4)$$

so that each X_k^q has the same law as $\max_{x \in [\bar{\sigma}^L, \sigma^L]_{\epsilon\mathbb{Z}}} \text{dist}(x, \bar{\partial}_\epsilon[\bar{\sigma}^L, \sigma^L]; \mathcal{G}^\epsilon|_{[\bar{\sigma}^L, \sigma^L]})$.

Define the sets

$$U_\epsilon := \bigcup_{q \in \{L, R\}} \bigcup_{k=1}^{K^q} [\bar{\sigma}_k^q, \sigma_k^q]_{\epsilon\mathbb{Z}} \quad (8.5)$$

and

$$A_\epsilon := \{z \in (0, \epsilon^{-u}]_{\mathbb{Z}} : (z-1, z]_{\epsilon\mathbb{Z}} \not\subset U_\epsilon\}. \quad (8.6)$$

By Lemma 8.5,

$$\mathbb{E}[\#A_\epsilon] \leq \sum_{z \in (0, \epsilon^{-u}]_{\mathbb{Z}}} z^{-2/\gamma^2} \leq \epsilon^{-u(1-2/\gamma^2)}. \quad (8.7)$$

Also define events

$$E_\epsilon := \{\text{diam}(\mathcal{G}^\epsilon|_{[x-1, x]}) \leq \epsilon^{-\chi-\zeta}, \forall x \in (0, \epsilon^{-u}]_{\epsilon\mathbb{Z}}\} \quad (8.8)$$

and

$$H_\epsilon := \{\sigma_{K^q}^q \leq \epsilon^{-v}, \forall q \in \{L, R\}\}. \quad (8.9)$$

By Theorem 1.15

$$\mathbb{P}[E_\epsilon^c] = o_\epsilon^\infty(\epsilon) \quad (8.10)$$

and by standard estimates for Brownian motion

$$\mathbb{P}[H_\epsilon^c] \leq \epsilon^{-(v-u)/2}. \quad (8.11)$$

We will now inductively define paths P_j which we will eventually concatenate to get a path from ϵ to $(\epsilon^{-u}, \infty]_{\epsilon\mathbb{Z}}$. Let $y_0 = \epsilon$ and let P_0 be the path in \mathcal{G}^ϵ consisting of the single cell y_0 . Inductively, suppose $j \in \mathbb{N}$ and $y_{j-1} \in (0, \epsilon^{-u}]_{\epsilon\mathbb{Z}}$ and a path P_{j-1} in \mathcal{G}^ϵ ending at a vertex of \mathcal{G}^ϵ adjacent to y_{j-1} have been defined. We consider three cases.

- If $y_{j-1} > \epsilon^{-u}$, we set $y_j = y_{j-1}$ and we let P_j be the empty path.
- If $y_{j-1} \leq \epsilon^{-u}$ but $y_j \notin U_\epsilon$ (defined in (8.5)), let $y_j := y_{j-1} + 1$ and let P_j be a path of minimal length from y_{j-1} to y_j in $\mathcal{G}^\epsilon|_{[y_{j-1}, y_j]}$. Note that on E_ϵ , we have $|P_j| \leq \epsilon^{-\chi-\zeta}$.
- If $y_{j-1} \in U_\epsilon$, let $q \in \{L, R\}$ and $k \in [1, K^q]_{\mathbb{Z}}$ be chosen such that $y_{j-1} \in [\bar{\sigma}_k^q, \sigma_k^q]_{\epsilon\mathbb{Z}}$ and $\sigma_k^q \geq \sigma_{k'}^{q'}$ for each $q' \in \{L, R\}$ and $k' \in \mathbb{N}$ with $y_{j-1} \in [\bar{\sigma}_{k'}^{q'}, \sigma_{k'}^{q'}]_{\epsilon\mathbb{Z}}$. Let x_j be the point of $\bar{\partial}_\epsilon[\bar{\sigma}_k^q, \sigma_k^q]$ which is closest to y_{j-1} in $\mathcal{G}^\epsilon|_{[\bar{\sigma}_k^q, \sigma_k^q]}$ and let P_j be a path from y_{j-1} to x_j with minimal length. Note that if the event H_ϵ of (8.9) occurs, then we have $|P_j| \leq X_k^q$, with X_k^q as in (8.4). Let $y_j \in [\sigma_k^q, \infty)_{\epsilon\mathbb{Z}}$ be chosen so that y_j is adjacent to x_j .

Let J be the smallest $j \in \mathbb{N}$ for which $y_j > \epsilon^{-u}$ and let P be the concatenation of the paths P_j for $j \in [0, J]_{\mathbb{Z}}$. Then P is a path in $\mathcal{G}^\epsilon|_{(0, \infty)}$ from ϵ to $y_J \in (\epsilon^{-u}, \infty)_{\epsilon\mathbb{Z}}$.

The above definition of the vertices y_j implies that $y_0 < y_1 < \dots < y_J$ and there are no two distinct integers $j, j' \in [0, J]_{\mathbb{Z}}$ for which either $y_j, y_{j'} \in (z-1, z]_{\epsilon\mathbb{Z}} \cap U_\epsilon$ for some $z \in \mathbb{Z}$ or $y_j, y_{j'} \in [\tilde{\sigma}_k^q, \sigma_k^q]_{\epsilon\mathbb{Z}}$ for some $q \in \{L, R\}$ and $k \in \mathbb{N}$. We therefore have

$$|P| \mathbb{1}_{E_\epsilon \cap H_\epsilon} \leq \sum_{q \in \{L, R\}} \sum_{k=1}^{K^q} X_k^q + \epsilon^{-\chi-\zeta} \#A_\epsilon, \quad (8.12)$$

with A_ϵ as in (8.6). Since the terminal point of P belongs to $(\epsilon^{-u}, \infty)_{\epsilon\mathbb{Z}}$, it follows that

$$\text{dist}(\epsilon, \bar{\partial}_\epsilon(0, \epsilon^{-u}); \mathcal{G}^\epsilon|_{(0, \epsilon^{-u}]}) \leq |P|.$$

By Proposition 7.1, it holds with probability at least $\epsilon^{o_\epsilon(1)}$ that $|P| \geq \epsilon^{-(\chi+\zeta)(1+u)}$. By (8.7), (8.10), and (8.11), we can take expectations of both sides of (8.12) to obtain

$$\epsilon^{-(\chi+\zeta)(1+u)+o_\epsilon(1)} \leq \mathbb{E} \left[\sum_{q \in \{L, R\}} \sum_{k=1}^{K^q} X_k^q \right] + \epsilon^{-\chi-\zeta-u(1-2/\gamma^2)+o_\epsilon(1)}.$$

Since we have assumed that $\chi > 1 - 2/\gamma^2$ (see the discussion right after the statement of Proposition 8.3) and ζ is arbitrary, we can send $\zeta \rightarrow 0$ and re-arrange to get

$$\mathbb{E} \left[\sum_{q \in \{L, R\}} \sum_{k=1}^{K^q} X_k^q \right] \geq \epsilon^{-\chi(1+u)+o_\epsilon(1)}.$$

Since each interval $[\tilde{\sigma}_k^q, \sigma_k^q]$ has length at least 1, we have $K^q \leq \epsilon^{-u}$ for each $q \in \{L, R\}$. Since the random variables X_k^q all have the same law (recall Lemma 8.4),

$$\mathbb{E}[X_k^q] \geq \epsilon^{-\chi+(1-\chi)u+o_\epsilon(1)}, \quad \forall q \in \{L, R\}, \forall k \in \mathbb{N}.$$

Since $u \in (0, v)$ is arbitrary (and independent of the definition of X_k^q), this implies the statement of the lemma. \square

Proof of Proposition 8.3. Let $u > v > 0$ and let

$$X := \max_{x \in [\tilde{\sigma}^L, \sigma^L]_{\epsilon\mathbb{Z}}} \text{dist}(x, \bar{\partial}_\epsilon[\tilde{\sigma}^L, \sigma^L]; \mathcal{G}^\epsilon|_{[\tilde{\sigma}^L, \sigma^L]}).$$

By Lemma 8.6,

$$\mathbb{E}[X \mathbb{1}_{(\sigma^L \leq \epsilon^{-v})}] \geq \epsilon^{-\chi+o_\epsilon(1)}.$$

By Theorem 1.15, it holds except on an event of probability $1 - o_\epsilon^\infty(\epsilon)$ that $X \mathbb{1}_{(\sigma^L \leq \epsilon^{-v})} \leq \epsilon^{-\chi(1+v)+o_\epsilon(1)}$. Since $v < u$, it follows that

$$\mathbb{P}[X \geq \epsilon^{-\chi+u}, \sigma^L \leq \epsilon^{-u}] \geq \mathbb{P}[X \geq \epsilon^{-\chi+u}, \sigma^L \leq \epsilon^{-v}] \geq \epsilon^{\chi v+o_\epsilon(1)}.$$

Since v is arbitrary we infer that

$$\mathbb{P}[X \geq \epsilon^{-\chi+u}, \sigma^L \leq \epsilon^{-u}] \geq \epsilon^{o_\epsilon(1)}. \quad (8.13)$$

Now let X' be defined in the same manner as X but with $\underline{\partial}_\epsilon[\tilde{\sigma}^L, \sigma^L]$ in place of $\bar{\partial}_\epsilon[\tilde{\sigma}^L, \sigma^L]$. Then (8.13) with X' in place of X is equivalent to (8.1). We will now argue that we can replace X by X' in (8.13). Indeed, the random variables X and X' each depend only on $(Z - Z_{\sigma^L})|_{[\tilde{\sigma}^L, \sigma^L]}$, and X' is obtained from this path in the same manner that X is obtained from its time reversal. The definitions of σ^L and $\tilde{\sigma}^L$ imply that the conditional law of $(Z - Z_{\sigma^L})|_{[\tilde{\sigma}^L, \sigma^L]}$ given σ^L and $\tilde{\sigma}^L$ is the same as the conditional law of $Z|_{[0, \sigma^L - \tilde{\sigma}^L]}$ given that $\underline{\Delta}_{[0, \sigma^L - \tilde{\sigma}^L]}^L = \bar{\Delta}_{[0, \sigma^L - \tilde{\sigma}^L]}^L = 0$, in the notation of Definition 2.2. This conditional law is invariant under time reversal. Consequently (8.13) for X implies (8.13) with X' in place of X . \square

8.2 Proof of Proposition 8.1

In this subsection we will deduce Proposition 8.1 from Proposition 8.3. To accomplish this, we will eventually define stopping times $\{S_j\}_{j \in \mathbb{N}_0}$ and $\{T_j\}_{j \in \mathbb{N}_0}$ for Z with $S_0 \leq T_0 \leq S_1 \leq T_1 \leq \dots$. These stopping times will satisfy (roughly speaking) the following conditions.

1. It is very unlikely that $S_j - T_{j-1}$ is much bigger than $\epsilon^{o_\epsilon(1)}$.
2. We have $\mathbb{P}[T_j - S_j \geq t] = \epsilon^{o_\epsilon(1)} t^{-1/2}$.
3. If $y \in [S_j, T_j]_{\epsilon\mathbb{Z}}$ for some $j \in \mathbb{N}$, then there is an $x \in (0, y]_{\epsilon\mathbb{Z}}$ with

$$\text{dist}\left(x, \partial_\epsilon(0, y] \cup \bar{\partial}_\epsilon^L(0, y]; \mathcal{G}^\epsilon|_{(0, y]}\right) \geq \epsilon^{-\chi+u}. \quad (8.14)$$

Conditions 1 and 2 and comparison to a sum of iid stable random variables will imply that for each fixed $v > 0$, it holds with high probability that most points of $(0, \epsilon^{-v}]_{\epsilon\mathbb{Z}}$ belong to one of the intervals $[S_j, T_j]_{\epsilon\mathbb{Z}}$ (see Lemma 8.9). By condition 3, our desired distance estimate holds with 1 replaced by a point y sampled uniformly at random from $(0, \epsilon^{-v}]_{\epsilon\mathbb{Z}}$. A short argument then allows us to transfer an estimate for a uniformly random y to an estimate for $y = 1$.

The times S_j will be defined using a slightly modified version of the event of Proposition 8.3. Our next lemma gives an estimate for the probability of this event.

Lemma 8.7. *Let σ^L and $\tilde{\sigma}^L$ be as in Proposition 8.3. Also fix $u, \zeta > 0$. For $\epsilon \in (0, 1)$, let $A^\epsilon = A^\epsilon(u)$ be the set of $x \in \bar{\partial}_\epsilon^R[\tilde{\sigma}^L, \sigma^L]_{\epsilon\mathbb{Z}}$ such that*

$$\inf_{t \in [x, \sigma^L]} (R_t - R_{\sigma^L}) \geq -\epsilon^\zeta.$$

For each $\epsilon \in (0, 1)$, we have

$$\mathbb{P}\left[\max_{x \in [\tilde{\sigma}^L, \sigma^L]_{\epsilon\mathbb{Z}}} \text{dist}(x, \partial_\epsilon[\tilde{\sigma}^L, \sigma^L] \cup A^\epsilon; \mathcal{G}^\epsilon|_{[\tilde{\sigma}^L, \sigma^L]}) \geq \epsilon^{-\chi+u}, \sigma^L \leq \epsilon^{-u}\right] \geq \epsilon^{o_\epsilon(1)}. \quad (8.15)$$

The set A^ϵ of Lemma 8.7 is the same as the set of $x \in \bar{\partial}_\epsilon^R[\tilde{\sigma}^L, \sigma^L]$ whose corresponding cell $\eta([x - \epsilon, x])$ intersects the length- ϵ^ζ arc of the upper right boundary of $\eta([\tilde{\sigma}^L, \sigma^L])$ started from $\eta(\sigma^L)$ (recall Section 2.2.1). For the proof of Proposition 8.7, we first need a simple lemma about Brownian motion.

Lemma 8.8. *Let $u > 0$ and for $\epsilon \in (0, 1)$ let $\tau_\epsilon^R := \inf\{t \geq 0 : R_t \leq -\epsilon^\zeta\}$. For each $v > 0$, we have (in the notation of Definition 2.2)*

$$\mathbb{P}\left[\tau_\epsilon^R \geq \epsilon^{2\zeta-v} \mid \underline{\Delta}_{(0,1]}^L = \bar{\Delta}_{(0,1]}^L = 0\right] \preceq \epsilon^{v/2} \quad (8.16)$$

with the implicit constant independent of ϵ .

Proof. This is obvious since when $\gamma = \sqrt{2}$, the coordinates L and R are independent, so the conditioning in (8.16) does not affect the law of R . \square

Proof of Lemma 8.7. Fix $v \in (0, u \wedge \zeta)$ (which we will eventually send to 0). Also let

$$\bar{\tau}_\epsilon^R := \sup\{t \leq \sigma^L : (R_t - R_{\sigma^L}) \leq -\epsilon^\zeta\}.$$

Recall that the conditional law of $(Z - Z_{\sigma^L})|_{[\tilde{\sigma}^L, \sigma^L]}$ given σ^L and $\tilde{\sigma}^L$ is the same as the conditional law of $Z|_{[0, \sigma^L - \tilde{\sigma}^L]}$ given that $\Delta_{[0, \sigma^L - \tilde{\sigma}^L]}^L = \bar{\Delta}_{[0, \sigma^L - \tilde{\sigma}^L]}^L = 0$. Since this law is invariant under time reversal and obeys Brownian scaling, we infer from Lemma 8.8 that

$$\mathbb{P}[\sigma^L - \bar{\tau}_\epsilon^R \geq \epsilon^{2u-v} \mid (\sigma^L, \tilde{\sigma}^L)] \preceq \epsilon^{v/2}.$$

By the upper bound in Theorem 1.15,

$$\mathbb{P}\left[\text{diam}\left(\mathcal{G}^\epsilon|_{[\bar{\tau}_\epsilon^R, \sigma^L]}\right) \geq \epsilon^{-\chi(1-2\zeta)-v}, \sigma^L \leq \epsilon^{-v} \mid (\sigma^L, \tilde{\sigma}^L)\right] \leq \epsilon^{v/2}.$$

By combining this with Proposition 8.3 (applied with v in place of u), we find that on an event of probability at least $\epsilon^{o_\epsilon(1)}$,

$$\text{diam}\left(\mathcal{G}^\epsilon|_{[\bar{\tau}_\epsilon^R, \sigma^L]}\right) \leq \epsilon^{-\chi(1-2\zeta)-v}, \quad \sigma^L \leq \epsilon^{-v}, \quad (8.17)$$

and

$$\max_{x \in [\tilde{\sigma}^L, \sigma^L]_{\mathbb{Z}}} \text{dist}(x, \underline{\partial}_\epsilon[\tilde{\sigma}^L, \sigma^L]; \mathcal{G}^\epsilon|_{[\tilde{\sigma}^L, \sigma^L]}) \geq 2\epsilon^{-\chi+v}. \quad (8.18)$$

Suppose $v \in (0, u)$ is chosen sufficiently small that $\chi(1-2u)+v < \chi-v$. For such a choice of v , whenever (8.17) holds,

$$\max_{y \in A^\epsilon} \text{dist}(\bar{x}_\epsilon, y; \mathcal{G}^\epsilon|_{[\tilde{\sigma}^L, \sigma^L]}) \leq \epsilon^{-\chi+v}$$

where here \bar{x}_ϵ is the right endpoint of $[\tilde{\sigma}^L, \sigma^L]_{\mathbb{Z}}$. Since $\bar{\Delta}_{[\tilde{\sigma}^L, \sigma^L]}^L = 0$, \bar{x}_ϵ is adjacent to $\underline{\partial}_\epsilon[\tilde{\sigma}^L, \sigma^L]$ in $\mathcal{G}^\epsilon|_{[\tilde{\sigma}^L, \sigma^L]}$. By the triangle inequality, if also (8.18) holds then

$$\max_{x \in [\tilde{\sigma}^L, \sigma^L]_{\mathbb{Z}}} \text{dist}(x, \underline{\partial}_\epsilon[\tilde{\sigma}^L, \sigma^L] \cup A^\epsilon; \mathcal{G}^\epsilon|_{[\tilde{\sigma}^L, \sigma^L]}) \geq \epsilon^{-\chi+v}.$$

Since $v < u$, we conclude. \square

We will now define our stopping times S_j and T_j . Fix $u, \zeta \in (0, 1/100)$ and a small $\epsilon > 0$ with $\epsilon \ll u \wedge \zeta$. Let $S_0 = \tilde{S}_0 = T_0 = 0$. Inductively, suppose that $j \in \mathbb{N}$ and $S_{j-1}, \tilde{S}_{j-1}, T_{j-1} \geq 0$ have been defined. We will define times S_j, \tilde{S}_j , and T_j . We first inductively define times $\sigma_{j,k}^L$ and $\tilde{\sigma}_{j,k}^L$ for $k \in \mathbb{N}_0$. Let $\sigma_{j,0}^L = \tilde{\sigma}_{j,0}^L := T_{j-1}$. Inductively, if $j \in \mathbb{N}$ and $\sigma_{j,k-1}^L$ and $\tilde{\sigma}_{j,k-1}^L$ have been defined, we let $\sigma_{j,k}^L$ be the minimum of $\sigma_{j,k-1}^L + \epsilon^{-\zeta}$ and the smallest $t \geq \sigma_{j,k-1}^L + 1$ for which there exists $\tilde{t} \in [\sigma_{j,k-1}^L, t-1]$ such that

$$L_t = L_{\tilde{t}} = \inf_{s \in [\sigma_{j,k-1}^L, \tilde{t}]} L_s. \quad (8.19)$$

If $\sigma_{j,k}^L < \sigma_{j,k-1}^L + \epsilon^{-\zeta}$ we let $\tilde{\sigma}_{j,k}^L$ be the largest corresponding time \tilde{t} as in (8.19) and otherwise we let $\tilde{\sigma}_{j,k}^L = \sigma_{j,k}^L$. Note that the times $\sigma_{j,k}^L$ and $\tilde{\sigma}_{j,k}^L$ are defined in almost the same manner as the times σ_k^L and $\tilde{\sigma}_k^L$ of (8.2) except that we start from time T_{j-1} instead of from time 0 and we truncate at $\epsilon^{-\zeta}$.

Let $A_{j,k}$ be the set of $x \in [\tilde{\sigma}_{j,k}^L, \sigma_{j,k}^L]_{\mathbb{Z}}$ such that

$$\inf_{t \in [x, \sigma_{j,k}^L]} (R_t - R_{\sigma_{j,k}^L}) \geq -\epsilon^\zeta,$$

which is defined in the same manner as the set A^ϵ of Lemma 8.7. Let K_j be the smallest $k \in \mathbb{N}$ for which

$$\max_{x \in [\tilde{\sigma}_{j,k}^L, \sigma_{j,k}^L]_{\mathbb{Z}}} \text{dist}\left(x, \underline{\partial}_\epsilon[\tilde{\sigma}_{j,k}^L, \sigma_{j,k}^L] \cup A_{j,k}; \mathcal{G}^\epsilon|_{[\tilde{\sigma}_{j,k}^L, \sigma_{j,k}^L]}\right) \geq \epsilon^{-\chi+u} \quad \text{and} \quad \sigma_{j,k}^L - \sigma_{j,k-1}^L < \epsilon^{-\zeta},$$

i.e. the event of Lemma 8.7 occurs with $[\tilde{\sigma}_{j,k}^L, \sigma_{j,k}^L]$ in place of $[\tilde{\sigma}^L, \sigma^L]$ (except with $\epsilon^{-\zeta}$ instead of ϵ^{-u}).

Let $S_j := \sigma_{j,K_j}^L$ and $\tilde{S}_j := \tilde{\sigma}_{j,K_j}^L$. Also let

$$T_j := \inf\{t \geq S_j : R_t - R_{S_j} \leq -\epsilon^\zeta\}$$

so that T_j is the smallest $t > 0$ for which no cell $\eta([x - \epsilon, x])$ for $x \in A_{j,K_j}$ shares a non-trivial boundary arc with the outer boundary of $\eta((-\infty, T_j])$.

Some of the above definitions are illustrated in Figure 13.

By induction, each of the times S_j and T_j is a stopping time for Z , as is each of the intermediate times $\sigma_{j,k}^L$. Furthermore, $(Z - Z_{T_{j-1}})|_{[T_{j-1}, T_j]}$ are iid. We also have the following estimate, which says that for $v > 100\zeta$, typically most times in $[0, \epsilon^{-v}]$ belong to one of the intervals $[S_j, T_j]$ rather than one of the intervals $[T_{j-1}, S_j]$.

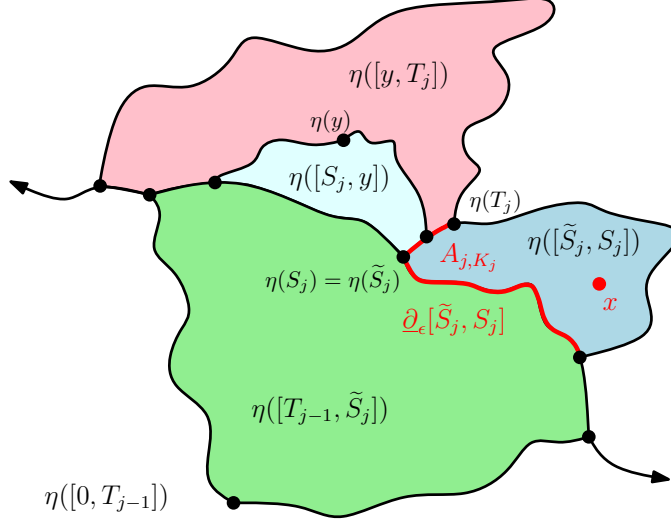


Figure 13: An illustration of the proof of Lemma 8.10. The space-filling SLE curve η traces the green region, then the blue region, then the light blue region, then the pink region. The black dots indicate the endpoints of the distinguished boundary arcs of each region (recall Section 2.2.1). The path $(Z - Z_{\tilde{S}_j})|_{[\tilde{S}_j, S_j]}$ has the same behavior as the path $(Z - Z_{\tilde{\sigma}^L})|_{[\tilde{\sigma}^L, \sigma^L]}$ on the event of Lemma 8.7. In particular, there is some $x \in [\tilde{S}_j, S_j]_{\epsilon\mathbb{Z}}$ which lies at $\mathcal{G}^\epsilon|_{[\tilde{S}_j, S_j]}$ -graph distance at least $\epsilon^{-\chi+u}$ from both red boundary arcs in the figure. The time T_j is (roughly speaking) the first time after S_j at which η hits part of $\bar{\partial}_\epsilon[\tilde{S}_j, S_j]$ which does not lie on the red arc A_{j,K_j} . If $y \in [S_j, T_j]_{\epsilon\mathbb{Z}}$, then any path in $\mathcal{G}^\epsilon|_{(0,y]}$ from x to $(0, y]_{\epsilon\mathbb{Z}} \setminus [\tilde{S}_j, S_j]_{\epsilon\mathbb{Z}}$ must pass through one of the red arcs, so the distance from x to $\partial_\epsilon(0, y] \cup \bar{\partial}_\epsilon^L(0, y]$ in $\mathcal{G}^\epsilon|_{(0,y]}$ is at least $\epsilon^{-\chi+u}$. On the other hand, Lemma 8.9 shows that with high probability, most points of $(0, \epsilon^{-v}]_{\epsilon\mathbb{Z}}$ lie in an interval of the form $[S_j, T_j]_{\epsilon\mathbb{Z}}$. This implies that if we sample y uniformly at random from $(0, \epsilon^{-v}]_{\epsilon\mathbb{Z}}$, then with high probability (8.14) holds.

Lemma 8.9. *Suppose $v \in (0, 1)$ with $v > 100\zeta$. Let j_* be the smallest $j \in \mathbb{N}$ for which $T_j \geq \epsilon^{-v}$. Then*

$$\mathbb{P}\left[\sum_{j=1}^{j_*} (S_j - T_{j-1}) > \epsilon^{-v/2-4\zeta}\right] = o_\epsilon^\infty(\epsilon).$$

Proof. By Lemma 8.7 (applied with $u \wedge \zeta$ in place of u), for each $j, k \in \mathbb{N}$ we have

$$\mathbb{P}[K_j = k \mid K_j > k-1] \geq \epsilon^{o_\epsilon(1)}.$$

Hence K_j is stochastically dominated by a geometric random variable with mean $\epsilon^{-o_\epsilon(1)}$. Since $\sigma_{j,k}^L - \sigma_{j,k-1}^L \leq \epsilon^{-\zeta}$ (by definition), it holds for each $\zeta > 0$ that

$$\mathbb{P}[S_j - T_{j-1} \geq \epsilon^{-2\zeta}] = o_\epsilon^\infty(\epsilon). \quad (8.20)$$

By standard estimates for Brownian motion, for $t > 0$ we have

$$\mathbb{P}[T_j - S_j \geq t] \asymp (\epsilon^\zeta t^{-1/2}) \wedge 1 \quad (8.21)$$

with universal implicit constant.

Let $\{Y_j\}_{j \in \mathbb{N}}$ be iid standard positive stable random variables with index $1/2$. It follows from (8.20) and (8.21) that we can find a constant $c > 0$ and a coupling of $\{Y_j\}_{j \in \mathbb{N}}$ with Z such that

$$T_j - T_{j-1} \geq c\epsilon^{2\zeta}Y_j, \quad \forall j \in \mathbb{N}.$$

For $m \in \mathbb{N}$ we have $\sum_{j=1}^m Y_j \stackrel{d}{=} m^2 Y_1$, so

$$\mathbb{P}[j_* > m] = \mathbb{P}[T_m \leq \epsilon^{-v}] \leq \mathbb{P}[Y_1 \leq c^{-1} m^{-2} \epsilon^{-v-2\zeta}].$$

Setting $m = \lfloor c^{-1/2} \epsilon^{-v/2-2\zeta} \rfloor$ and using, e.g., the fact that Y_1 has the same law as the first time a standard linear Brownian motion hits 1, we obtain

$$\mathbb{P}[j_* > \lfloor \epsilon^{-v/2-2\zeta} \rfloor] = o_\epsilon^\infty(\epsilon).$$

On the other hand, (8.20) implies that

$$\mathbb{P}\left[\sum_{j=1}^{\lfloor \epsilon^{-v/2-2\zeta} \rfloor} (S_j - T_{j-1}) \geq \epsilon^{-v/2-4\zeta}\right] = o_\epsilon^\infty(\epsilon),$$

which concludes the proof. \square

Next we prove a variant of Proposition 8.1 with a random sub-interval of $(0, \epsilon^{-v}]$ in place of $(0, 1]$.

Lemma 8.10. *Let $v \in (100\zeta, 1)$ and let y be sampled uniformly from $(0, \epsilon^{-v}]_{\epsilon\mathbb{Z}}$, independently from Z . For each $u > 0$, we have*

$$\mathbb{P}\left[\max_{x \in (0, y]_{\epsilon\mathbb{Z}}} \text{dist}\left(x, \underline{\partial}_\epsilon(0, y] \cup \bar{\partial}_\epsilon^L(0, y]; \mathcal{G}^\epsilon|_{(0, y]}\right) \geq \epsilon^{-\chi+u}\right] \geq 1 - \epsilon^{v/2+o_\epsilon(1)}. \quad (8.22)$$

Proof. See Figure 13 for an illustration of the proof. Define the times S_j , \tilde{S}_j , and T_j for $j \in \mathbb{N}_0$ as above and the time j_* as in Lemma 8.9. We claim that if there exists $j \in [1, j_*]_{\mathbb{Z}}$ for which $y \in [S_j, T_j]_{\epsilon\mathbb{Z}}$, then the event in (8.22) holds. Indeed, suppose such a j exists. By definition of S_j and \tilde{S}_j , there exists $x \in [\tilde{S}_j, S_j]_{\epsilon\mathbb{Z}}$ such that

$$\text{dist}\left(x, \underline{\partial}_\epsilon[\tilde{S}_j, S_j] \cup A_{j, K_j}; \mathcal{G}^\epsilon|_{[\tilde{S}_j, S_j]}\right) \geq \epsilon^{-\chi+u}.$$

Since $y \in [S_j, T_j]_{\epsilon\mathbb{Z}}$, the vertex set $(S_j, x]_{\epsilon\mathbb{Z}}$ is not adjacent in \mathcal{G}^ϵ to any element of $\partial_\epsilon[\tilde{S}_j, S_j]$ which lies outside of $\underline{\partial}_\epsilon[\tilde{S}_j, S_j] \cup A_{j, K_j}$. Clearly, $(0, \tilde{S}_j)_{\epsilon\mathbb{Z}}$ is not incident to any element of $\bar{\partial}_\epsilon[\tilde{S}_j, S_j]$. Consequently, any path from x to an element of $(0, y]_{\epsilon\mathbb{Z}} \setminus [\tilde{S}_j, S_j]_{\epsilon\mathbb{Z}}$ must pass through $\underline{\partial}_\epsilon[\tilde{S}_j, S_j] \cup A_{j, K_j}$, so must have length at least $\epsilon^{-\chi+u}$. Since $\bar{\Delta}_{[\tilde{S}_j, S_j]}^L = 0$, $\bar{\partial}_\epsilon^L(0, y]$ contains no elements of $[\tilde{S}_j, S_j]_{\epsilon\mathbb{Z}}$. We thus obtain our claim.

Consequently, we only need to prove a lower bound for

$$\mathbb{P}[\exists j \in [1, j_*]_{\mathbb{Z}} \text{ for which } y \in [S_j, T_j]_{\epsilon\mathbb{Z}}].$$

By Lemma 8.9, this probability is at least $1 - \epsilon^{v/2-4\zeta} - o_\epsilon^\infty(\epsilon)$. Since ζ is arbitrary, we obtain the statement of the lemma. \square

Proof of Proposition 8.1. Fix $v, u' \in (0, 1)$. By Lemma 8.10 (with $\epsilon^{\frac{1-v}{1+v}}$ in place of ϵ and u' in place of u) and scale invariance we can find a *deterministic* $y \in (0, \epsilon^v]_{\epsilon\mathbb{Z}}$ for which

$$\mathbb{P}\left[\max_{x \in (0, y]_{\epsilon\mathbb{Z}}} \text{dist}\left(x, \underline{\partial}_\epsilon(0, y] \cup \bar{\partial}_\epsilon^L(0, y]; \mathcal{G}^\epsilon|_{(0, y]}\right) \geq \epsilon^{-\frac{1-v}{1+v}(\chi-u')}\right] \geq 1 - \epsilon^{\frac{v-v^2}{2(1+v)}+o_\epsilon(1)}.$$

By translation invariance, the same estimate holds with $(1-y, 1]_{\epsilon\mathbb{Z}}$ in place of $(0, y]_{\epsilon\mathbb{Z}}$. Furthermore, if

$$\text{dist}\left(x, \underline{\partial}_\epsilon(1-y, 1] \cup \bar{\partial}_\epsilon^L(1-y, 1]; \mathcal{G}^\epsilon|_{(1-y, 1]}\right) \geq \epsilon^{-\frac{1-v}{2(1+v)}(\chi-u')}$$

then also

$$\text{dist}\left(x, \underline{\partial}_\epsilon(0, 1] \cup \bar{\partial}_\epsilon^L(0, 1]; \mathcal{G}^\epsilon|_{(0, 1]}\right) \geq \epsilon^{-\frac{1-v}{2(1+v)}(\chi-u')}$$

since any path from x to $(0, 1-y]_{\epsilon\mathbb{Z}}$ must pass through $\underline{\partial}_\epsilon(1-y, 1]$. Now choose v in such a way that $\frac{1-v}{1+v}(\chi-u') = \chi-u$, so that $v \geq cu + o_{u'}(1)$ for a constant $c > 0$ depending only on χ . Then the above discussion implies that

$$\mathbb{P}\left[\max_{x \in (0, 1]_{\epsilon\mathbb{Z}}} \text{dist}\left(x, \underline{\partial}_\epsilon(0, 1] \cup \bar{\partial}_\epsilon^L(0, 1]; \mathcal{G}^\epsilon|_{(0, 1]}\right) \geq \epsilon^{-\chi+u}\right] \geq 1 - \epsilon^{cu+o_{u'}(1)+o_\epsilon(1)}.$$

We conclude by sending $u' \rightarrow 0$. \square

References

- [AB14] J. Ambjørn and T. G. Budd. Geodesic distances in Liouville quantum gravity. *Nuclear Physics B*, 889:676–691, December 2014, 1405.3424.
- [AK14] S. Andres and N. Kajino. Continuity and estimates of the Liouville heat kernel with applications to spectral dimensions. *ArXiv e-prints*, July 2014, 1407.3240.
- [Ald91a] D. Aldous. The continuum random tree. I. *Ann. Probab.*, 19(1):1–28, 1991. MR1085326 (91i:60024)
- [Ald91b] D. Aldous. The continuum random tree. II. An overview. In *Stochastic analysis (Durham, 1990)*, volume 167 of *London Math. Soc. Lecture Note Ser.*, pages 23–70. Cambridge Univ. Press, Cambridge, 1991. MR1166406 (93f:60010)
- [Ald93] D. Aldous. The continuum random tree. III. *Ann. Probab.*, 21(1):248–289, 1993. MR1207226 (94c:60015)
- [ANR⁺98] J. Ambjørn, J. L. Nielsen, J. Rolf, D. Boulatov, and Y. Watabiki. The spectral dimension of 2D quantum gravity. *Journal of High Energy Physics*, 2:010, February 1998, hep-th/9801099.
- [BBI01] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. MR1835418
- [Ber07] O. Bernardi. Bijective counting of tree-rooted maps and shuffles of parenthesis systems. *Electron. J. Combin.*, 14(1):Research Paper 9, 36 pp. (electronic), 2007. MR2285813 (2007m:05125)
- [Ber08] O. Bernardi. Tutte polynomial, subgraphs, orientations and sandpile model: new connections via embeddings. *Electron. J. Combin.*, 15(1):Research Paper 109, 53, 2008, math/0612003. MR2438581 (2009f:05110)
- [Ber13] N. Berestycki. Diffusion in planar Liouville quantum gravity. *ArXiv e-prints*, January 2013, 1301.3356.
- [BLR15] N. Berestycki, B. Laslier, and G. Ray. Critical exponents on Fortuin–Kasteleyn weighted planar maps. *ArXiv e-prints*, February 2015, 1502.00450.
- [BM15] J. Bettinelli and G. Miermont. Compact Brownian surfaces I. Brownian disks. *ArXiv e-prints*, July 2015, 1507.08776.
- [BMR16] E. Baur, G. Miermont, and G. Ray. Classification of scaling limits of uniform quadrangulations with a boundary. *ArXiv e-prints*, August 2016, 1608.01129.
- [BS01] I. Benjamini and O. Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:no. 23, 13 pp. (electronic), 2001, 0011019. MR1873300 (2002m:82025)
- [Che15] L. Chen. Basic properties of the infinite critical-FK random map. *ArXiv e-prints*, February 2015, 1502.01013.
- [CL14] N. Curien and J.-F. Le Gall. The Brownian plane. *J. Theoret. Probab.*, 27(4):1249–1291, 2014, 1204.5921. MR3278940
- [dBE52] N. G. de Bruijn and P. Erdős. Some linear and some quadratic recursion formulas. II. *Nederl. Akad. Wetensch. Proc. Ser. A*. **55** = *Indagationes Math.*, 14:152–163, 1952. MR0047162
- [DD16] J. Ding and A. Dunlap. Liouville first-passage percolation: subsequential scaling limit at high temperature. *ArXiv e-prints*, May 2016, 1605.04011.
- [DG15] J. Ding and S. Goswami. First passage percolation on the exponential of two-dimensional branching random walk. *ArXiv e-prints*, November 2015, 1511.06932.

- [DG16a] J. Ding and S. Goswami. Liouville first passage percolation: the weight exponent is strictly less than 1 at high temperatures. *ArXiv e-prints*, May 2016, 1605.08392.
- [DG16b] J. Ding and S. Goswami. Upper bounds on Liouville first passage percolation and Watabiki’s prediction. *ArXiv e-prints*, October 2016, 1610.09998.
- [DIM77] R. T. Durrett, D. L. Iglehart, and D. R. Miller. Weak convergence to Brownian meander and Brownian excursion. *Ann. Probability*, 5(1):117–129, 1977. MR0436353 (55 #9300)
- [DMS14] B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees. *ArXiv e-prints*, September 2014, 1409.7055.
- [DS11] B. Duplantier and S. Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011, 1206.0212. MR2819163 (2012f:81251)
- [DW15a] D. Denisov and V. Wachtel. Random walks in cones. *Ann. Probab.*, 43(3):992–1044, 2015, 1110.1254. MR3342657
- [DW15b] J. Duraj and V. Wachtel. Invariance principles for random walks in cones. *ArXiv e-prints*, August 2015, 1508.07966.
- [DZ15] J. Ding and F. Zhang. Non-universality for first passage percolation on the exponential of log-correlated Gaussian fields. *ArXiv e-prints*, June 2015, 1506.03293.
- [DZ16] J. Ding and F. Zhang. Liouville first passage percolation: geodesic dimension is strictly larger than 1 at high temperatures. *ArXiv e-prints*, October 2016, 1610.02766.
- [Eva85] S. N. Evans. On the Hausdorff dimension of Brownian cone points. *Math. Proc. Cambridge Philos. Soc.*, 98(2):343–353, 1985. MR795899 (86j:60185)
- [GHM15] E. Gwynne, N. Holden, and J. Miller. An almost sure KPZ relation for SLE and Brownian motion. *ArXiv e-prints*, December 2015, 1512.01223.
- [GHMS16] E. Gwynne, N. Holden, J. Miller, and X. Sun. Brownian motion correlation in the peanosphere for $\kappa > 8$. *Annales de l’Institut Henri Poincaré*, 2016, 1510.04687. To appear.
- [GHS16] E. Gwynne, N. Holden, and X. Sun. Joint scaling limit of a bipolar-oriented triangulation and its dual in the peanosphere sense. *ArXiv e-prints*, March 2016, 1603.01194.
- [GKMW16] E. Gwynne, A. Kassel, J. Miller, and D. B. Wilson. Active spanning trees with bending energy on planar maps and SLE-decorated Liouville quantum gravity for $\kappa \geq 8$. *ArXiv e-prints*, March 2016, 1603.09722.
- [GM16a] E. Gwynne and J. Miller. Convergence of the topology of critical Fortuin-Kasteleyn planar maps to that of CLE_κ on a Liouville quantum surface. In preparation, 2016.
- [GM16b] E. Gwynne and J. Miller. Scaling limit of the uniform infinite half-plane quadrangulation in the Gromov-Hausdorff-Prokhorov-uniform topology. *ArXiv e-prints*, August 2016, 1608.00954.
- [GMS15] E. Gwynne, C. Mao, and X. Sun. Scaling limits for the critical Fortuin-Kasteleyn model on a random planar map I: cone times. *ArXiv e-prints*, February 2015, 1502.00546.
- [GRV13a] C. Garban, R. Rhodes, and V. Vargas. Liouville Brownian motion. *ArXiv e-prints*, January 2013, 1301.2876.
- [GRV13b] C. Garban, R. Rhodes, and V. Vargas. On the heat kernel and the Dirichlet form of Liouville Brownian Motion. *ArXiv e-prints*, February 2013, 1302.6050.
- [GS15a] E. Gwynne and X. Sun. Scaling limits for the critical Fortuin-Kasteleyn model on a random planar map II: local estimates and empty reduced word exponent. *ArXiv e-prints*, May 2015, 1505.03375.

- [GS15b] E. Gwynne and X. Sun. Scaling limits for the critical Fortuin-Kasteleyn model on a random planar map III: finite volume case. *ArXiv e-prints*, October 2015, 1510.06346.
- [HS16] N. Holden and X. Sun. SLE as a mating of trees in Euclidean geometry. *ArXiv e-prints*, October 2016, 1610.05272.
- [KMSW15] R. Kenyon, J. Miller, S. Sheffield, and D. B. Wilson. Bipolar orientations on planar maps and SLE_{12} . *ArXiv e-prints*, November 2015, 1511.04068.
- [KN04] W. Kager and B. Nienhuis. A guide to stochastic Löwner evolution and its applications. *Journal of Statistical Physics*, 115(5-6):1149–1229, 2004, math-ph/0312056.
- [Law05] G. F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. MR2129588 (2006i:60003)
- [Law15] G. F. Lawler. Minkowski content of the intersection of a Schramm-Loewner evolution (SLE) curve with the real line. *J. Math. Soc. Japan*, 67(4):1631–1669, 2015. MR3417507
- [Le 05] J.-F. Le Gall. Random trees and applications. *Probab. Surv.*, 2:245–311, 2005, math/0511515. MR2203728 (2007h:60078)
- [Le 07] J.-F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169(3):621–670, 2007, math/0607567. MR2336042 (2008i:60022)
- [Le 13] J.-F. Le Gall. Uniqueness and universality of the Brownian map. *Ann. Probab.*, 41(4):2880–2960, 2013, 1105.4842. MR3112934
- [Le 14] J.-F. Le Gall. Random geometry on the sphere. *ArXiv e-prints*, March 2014, 1403.7943.
- [LW06] J.-F. Le Gall and M. Weill. Conditioned Brownian trees. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(4):455–489, 2006, math/0501066. MR2242956 (2007k:60268)
- [Mie09] G. Miermont. Random maps and their scaling limits. In *Fractal geometry and stochastics IV*, volume 61 of *Progr. Probab.*, pages 197–224. Birkhäuser Verlag, Basel, 2009. MR2762678 (2012a:60017)
- [Mie13] G. Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.*, 210(2):319–401, 2013, 1104.1606. MR3070569
- [MM06] J.-F. Marckert and A. Mokkadem. Limit of normalized quadrangulations: the Brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006. MR2294979
- [Moo28] R. L. Moore. Concerning upper semi-continuous collections of continua. *Trans. Amer. Math. Soc.*, 27(4):416–428, 1928.
- [MRVZ14] P. Maillard, R. Rhodes, V. Vargas, and O. Zeitouni. Liouville heat kernel: regularity and bounds. *ArXiv e-prints*, June 2014, 1406.0491.
- [MS13] J. Miller and S. Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. *ArXiv e-prints*, February 2013, 1302.4738.
- [MS15a] J. Miller and S. Sheffield. An axiomatic characterization of the Brownian map. *ArXiv e-prints*, June 2015, 1506.03806.
- [MS15b] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map I: The $QLE(8/3,0)$ metric. *ArXiv e-prints*, July 2015, 1507.00719.
- [MS15c] J. Miller and S. Sheffield. Liouville quantum gravity spheres as matings of finite-diameter trees. *ArXiv e-prints*, June 2015, 1506.03804.

- [MS16a] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding. *ArXiv e-prints*, May 2016, 1605.03563.
- [MS16b] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map III: the conformal structure is determined. *ArXiv e-prints*, August 2016, 1608.05391.
- [MS16c] J. Miller and S. Sheffield. Quantum Loewner Evolution. *Duke Mathematics Journal*, to appear, 2016, 1312.5745.
- [MS16d] J. Miller and S. Sheffield. Imaginary geometry I: interacting SLEs. *Probab. Theory Related Fields*, 164(3-4):553–705, 2016, 1201.1496. MR3477777
- [Mul67] R. C. Mullin. On the enumeration of tree-rooted maps. *Canad. J. Math.*, 19:174–183, 1967. MR0205882 (34 #5708)
- [RV14] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and applications: A review. *Probab. Surv.*, 11:315–392, 2014, 1305.6221. MR3274356
- [Sch00] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000, math/9904022. MR1776084 (2001m:60227)
- [She07] S. Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139(3-4):521–541, 2007. MR2322706 (2008d:60120)
- [She16a] S. Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Annals of Probability*, to appear, 2016, 1012.4797.
- [She16b] S. Sheffield. Quantum gravity and inventory accumulation. *Annals of Probability*, to appear, 2016, 1108.2241.
- [Shi85] M. Shimura. Excursions in a cone for two-dimensional Brownian motion. *J. Math. Kyoto Univ.*, 25(3):433–443, 1985. MR807490 (87a:60095)
- [SS13] O. Schramm and S. Sheffield. A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):47–80, 2013, math/0605337. MR3101840
- [Wat93] Y. Watabiki. Analytic study of fractal structure of quantized surface in two-dimensional quantum gravity. *Progr. Theoret. Phys. Suppl.*, (114):1–17, 1993. Quantum gravity (Kyoto, 1992).
- [Zha08] D. Zhan. Duality of chordal SLE. *Invent. Math.*, 174(2):309–353, 2008, 0712.0332. MR2439609 (2010f:60239)